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Solution Concepts and Well-posedness of Hybrid Systems

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HYCON Summer School on Hybrid Systems
Key issues:

- Solution concepts
- Well-posedness: existence & uniqueness of solutions given an initial condition

Outline lecture

- Smooth systems: differential equations
- Switched systems: Discontinuous differential equations: “classics”
- Hybrid automata
- Zenoness: importance of choice of solution concept
- Some piecewise linear, linear relay and complementarity systems
- Summary
Solution concept

Description format / syntax / model

↓
solutions / trajectories / executions/ semantics/ behavior

Well-posedness: given initial condition does there exist a solution and is it unique?

Let’s start simple...
Smooth differential equations

**Example** \( \dot{x} = f(t, x) \quad x(t_0) = x_0. \)

A solution trajectory is a function \( x : [t_0, t_1] \mapsto \mathbb{R}^n \) that is continuous, differentiable and satisfies \( x(t_0) = x_0 \) and

\[
\dot{x}(t) = f(t, x(t)) \text{ for all } t \in (t_0, t_1)
\]

**Well-posedness:** given initial condition does there exists a solution and is it unique?
Well-posedness

Example \( \dot{x} = 2\sqrt{x}, \ x(0) = 0 \). Solutions: \( x(t) = 0 \) and \( x(t) = t^2 \).

Local existence and uniqueness of solutions given an initial condition:

**Theorem 1** Let \( f(t, x) \) be piecewise continuous in \( t \) and satisfy the following Lipschitz condition: there exist an \( L > 0 \) and \( r > 0 \) such that

\[
\|f(t, x) - f(t, y)\| \leq L\|x - y\|
\]

and all \( x \) and \( y \) in a neighborhood \( B := \{x \in \mathbb{R}^n \mid \|x - x_0\| < r\} \) of \( x_0 \) and for all \( t \in [t_0, t_1] \).

\[
\Downarrow
\]

There is a \( \delta > 0 \) s.t. a unique solution exists on \([t_0, t_0 + \delta]\) starting in \( x_0 \) at \( t_0 \).
Global well-posedness

Example $\dot{x} = x^2 + 1, x(0) = 0$. Solution: $x(t) = \tan t$. **Local** on $[0, \pi/2)$.

- Note that we have $\lim_{t \uparrow \pi/2} x(t) = \infty$. Finite escape time!

Theorem 2 (Global Lipschitz condition) Suppose $f(t, x)$ is piecewise continuous in $t$ and satisfies

$$\|f(t, x) - f(t, y)\| \leq L\|x - y\|$$

for all $x, y$ in $\mathbb{R}^n$ and for all $t \in [t_0, t_1]$. Then, a unique solution exists on $[t_0, t_1]$ for any initial state $x_0$ at $t_0$.

- Not necessary: $\dot{x} = -x^3$ not glob. Lipsch., but unique global solutions.

- Also in hybrid systems, but even more awkward stuff (Zeno)
Discontinuous differential equations: a class of switched systems

\[ x' = f_+(x) \quad \text{if} \quad x \in C_+ := \{ x \in \mathbb{R}^n \mid \phi(x) > 0 \} \]
\[ x' = f_-(x) \quad \text{if} \quad x \in C_- := \{ x \in \mathbb{R}^n \mid \phi(x) < 0 \} \]

- \( x \) in interior of \( C_- \) or \( C_+ \): just follow!
- \( f_-(x) \) and \( f_+(x) \) point in same direction: just follow!

\[ n(x) = \frac{\nabla \phi(x)}{||\nabla \phi(x)||} \quad \text{then} \quad (n(x)^T f_-(x)) \cdot (n(x)^T f_+(x)) > 0 \]

- \( n(x)^T f_+(x) > 0 \) (\( f_+(x) \) points towards \( C_+ \)) and \( n(x)^T f_-(x) < 0 \) (\( f_-(x) \) points towards \( C_- \)):
  At least two trajectories
Sliding modes

\[ f(x_0) \]

\[ \phi(x) = 0 \]

\[ n(x)^T f_+(x) < 0 \quad (f_+(x) \text{ points towards } C_-) \quad \text{and} \quad n(x)^T f_-(x) > 0 \quad (f_-(x) \text{ points towards } C_+) \]

No classical solution

- Relaxation: spatial (hysteresis) \( \Delta \), time delay \( \tau \), smoothing \( \varepsilon \)
- Chattering / infinitely fast switching (limit case \( \Delta \downarrow 0, \varepsilon \downarrow 0, \text{ and } \tau \downarrow 0 \))

Filippov’s convex definition: convex combination of both dynamics

\[ \dot{x} = \lambda f_+(x) + (1 - \lambda) f_-(x) \quad \text{with} \quad 0 \leq \lambda \leq 1 \]

such that \( x \) moves (“slides”) along \( \phi(x) = 0 \). “Third mode ... ”
Differential inclusions

\[
\dot{x} = \begin{cases} 
  f_+(x), & \text{if } \phi(x) > 0 \\
  \lambda f_+(x) + (1 - \lambda) f_-(x), & \text{if } \phi(x) = 0, \ 0 \leq \lambda \leq 1 \\
  f_-(x), & \text{if } \phi(x) < 0,
\end{cases}
\]

Differential inclusion \( \dot{x} \in F(x) \) with set-valued

\[
F(x) = \begin{cases} 
  \{f_+(x)\}, & \phi(x) > 0 \\
  \{\lambda f_+(x) + (1 - \lambda) f_-(x) \mid \lambda \in [0, 1]\}, & \phi(x) = 0 \\
  \{f_-(x)\}, & \phi(x) < 0
\end{cases}
\]

Definition 3 A function \( x : [a, b] \mapsto \mathbb{R}^n \) is a solution of \( \dot{x} \in F(x) \), if \( x \) is absolutely continuous and satisfies \( \dot{x}(t) \in F(x(t)) \) for almost all \( t \in [a, b] \).
A well-posedness result

\[ x' = f_+(x) \quad \text{for} \quad C_+ \]

\[ x' = f_-(x) \quad \text{for} \quad C_- \]

\[ \phi(x) = 0 \]

- \( f_- \) and \( f_+ \) are continuously differentiable (\( C^1 \))
- \( \phi \) is \( C^2 \)
- the discontinuity vector \( h(x) := f_+(x) - f_-(x) \) is \( C^1 \)

If for each point \( x \) with \( \phi(x) = 0 \) at least one of the two inequalities \( n(x)^T f_+(x) < 0 \) or \( n(x)^T f_-(x) > 0 \) (for different points a different inequality may hold), then the Filippov solutions exist and are unique.
Alternative: Utkin’s equivalent control definition

\[ \dot{x} = f(x, u) \] with \( u = \begin{cases} g_+(x), & \xi(x) > 0 \\ g_-(x), & \xi(x) < 0 \end{cases} \)

- Sliding mode: \( f_+(x) := f(x, g_+(x)) \) and \( f_-(x) := f(x, g_-(x)) \) point outside \( C_+ \) and \( C_- \), resp.

\[ u_{\text{equiv}} \in U(x) := \begin{cases} \{g_+(x)\}, & \text{if } \xi(x) > 0 \\ \{\lambda g_+(x) + (1 - \lambda)g_-(x) \mid \lambda \in [0, 1]\}, & \text{if } \xi(x) = 0 \\ \{g_-(x)\}, & \text{if } \xi(x) < 0 \end{cases} \]

Differential inclusion

\[ \dot{x} \in F(x) := f(x, U(x)) = \{f(x, u) \mid u \in U(x)\} \]

“Idealization” determines Filippov/ Utkin / different solution concept!
Example

\[
\begin{align*}
\dot{x}_1 &= -x_1 + x_2 - u \\
\dot{x}_2 &= 2x_2(u^2 - u - 1) \\
u &= \begin{cases} 
1, & \text{if } x_1 > 0 \\
-1, & \text{if } x_1 < 0.
\end{cases}
\end{align*}
\]

Two “original” dynamics:

- **\( C_+ \):** \( x_1 > 0 \):
  \[
  \begin{align*}
  \dot{x}_1 &= -x_1 + x_2 - 1 \\
  \dot{x}_2 &= -2x_2
  \end{align*}
  \]

- **\( C_- \):** \( x_1 < 0 \):
  \[
  \begin{align*}
  \dot{x}_1 &= -x_1 + x_2 + 1 \\
  \dot{x}_2 &= 2x_2
  \end{align*}
  \]
Vector fields
Vector fields: zoom
Sliding modes?

Two “original” dynamics:

- $C_+: x_1 > 0$: \[ \dot{x} = f_+(x) \]
  \[ \dot{x}_1 = -x_1 + x_2 - 1 \]
  \[ \dot{x}_2 = -2x_2 \]

- $C_-: x_1 < 0$: \[ \dot{x} = f_-(x) \]
  \[ \dot{x}_1 = -x_1 + x_2 + 1 \]
  \[ \dot{x}_2 = 2x_2 \]

- $n(x)^T f_+(x) = x_2 - 1 < 0 \implies x_2 < 1$
- $n(x)^T f_-(x) = x_2 + 1 > 0 \implies x_2 > -1$
- Sliding possible in $x_1 = 0$ and $x_2 \in [-1, 1]$. 
Filippov’s solution concept

Two “original” dynamics:

- $C_+: x_1 > 0$: $\dot{x} = f_+(x)$
  \[
  \dot{x}_1 = -x_1 + x_2 - 1 \\
  \dot{x}_2 = -2x_2 
  \]

- $C_-: x_1 < 0$: $\dot{x} = f_-(x)$
  \[
  \dot{x}_1 = -x_1 + x_2 + 1 \\
  \dot{x}_2 = 2x_2 
  \]

- Filippov: Take convex combination of dynamics such that state slides on $x_1 = 0$: Hence, $x_1 = \dot{x}_1 = 0$.
  \[
  \lambda(x_2 - 1) + (1 - \lambda)(x_2 + 1) = 0 \text{ implies } \lambda = \frac{1}{2}(x_2 + 1) 
  \]
  \[
  \dot{x}_2 = \lambda(-2x_2) + (1 - \lambda)(2x_2) = -2x_2^2 
  \]
  \[
  0 \text{ is unstable equilibrium.} 
  \]
Vector fields: Filippov’s case
Utkin’s solution concept

\[
\dot{x}_1 = -x_1 + x_2 - u \\
\dot{x}_2 = 2x_2(u^2 - u - 1)
\]

\[
u = \begin{cases} 
1, & \text{if } x_1 > 0 \\
-1, & \text{if } x_1 < 0.
\end{cases}
\]

- The equivalent control $u_{\text{equiv}}$ is such that state slides along $x_1 = 0$. Hence, $x_1 = \dot{x}_1 = 0$ and thus $u_{\text{equiv}} = x_2$ and

\[
\dot{x}_2 = 2x_2(x_2^2 - x_2 - 1)
\]

- Equilibria: -0.618 (unstable) and 0 (stable)
Vector fields
Solution trajectories
Two relaxations

- **Smoothing** \( u(t) = \tanh(x_1/\varepsilon) \)
- **Hysteresis type of switching** parameter \( \Delta \)
Solution trajectories: Filippov’s case + hysteresis
Solution trajectories: Utkin’s case + smoothing
Conclusions on discontinuous dynamical systems

- Two mathematical solutions concepts: Filippov + Utkin
- Both limit cases ("idealizations") of very fast switching
- Which one you use depends on non-ideal cases (regularizations)
- Sliding mode might be seen as third mode in hybrid automaton. Some subtleties in HA solution concept!
From classical to modern solution concepts
Hybrid Systems

- Smooth phases (governed by differential equations)
- Discrete events and actions

Smooth phases separated by event times ...
Event times

\[\dot{x}_1(t) = x_3(t)\]
\[\dot{x}_2(t) = x_4(t)\]
\[\dot{x}_3(t) = -2x_1(t) + x_2(t) + z(t)\]
\[\dot{x}_4(t) = x_1(t) - x_2(t)\]
\[w(t) = x_1(t)\]
\[w(t) \geq 0, \ z(t) \geq 0, \ \{w(t) = 0 \text{ or } z(t) = 0\}\]

<table>
<thead>
<tr>
<th>unconstrained</th>
<th>constrained</th>
</tr>
</thead>
<tbody>
<tr>
<td>[\dot{x}_1(t) = x_3(t)]</td>
<td>[\dot{x}_1(t) = x_3(t)]</td>
</tr>
<tr>
<td>[\dot{x}_2(t) = x_4(t)]</td>
<td>[\dot{x}_2(t) = x_4(t)]</td>
</tr>
<tr>
<td>[\dot{x}_3(t) = -2x_1(t) + x_2(t)]</td>
<td>[\dot{x}_3(t) = -2x_1(t) + x_2(t) + z(t)]</td>
</tr>
<tr>
<td>[\dot{x}_4(t) = x_1(t) + x_2(t)]</td>
<td>[\dot{x}_4(t) = x_1(t) + x_2(t)]</td>
</tr>
<tr>
<td>[z(t) = 0]</td>
<td>[w(t) = x_1(t) = 0]</td>
</tr>
</tbody>
</table>

\[w(t) \geq 0\]
\[z(t) \geq 0\]
• Event times set $\mathcal{E}$ is $\{0, 1, 1 + \frac{\pi}{2}\}$
Example: Bouncing ball

- Reset $x_2(\tau^+) := -cx_2(\tau^-)$ when $x_1(\tau^-) = 0$ and $x_2(\tau^-) \leq 0$.
- The event times: $\tau_{i+1} = \tau_i + \frac{2c_i x_2(0)}{g}$ when $x_1(0) = 0$ and $x_2(0) > 0$.
- $\lim_{i \to \infty} \tau_i = \tau^* = \frac{2x_2(0)}{g-gc} < \infty$
Zeno of Elea and one of his paradoxes

Distance Travelled (m) by Achilles

<table>
<thead>
<tr>
<th>Event times of A reaching previous T position</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
</tr>
<tr>
<td>1.5</td>
</tr>
<tr>
<td>1.75</td>
</tr>
<tr>
<td>1.875</td>
</tr>
<tr>
<td>1.9375</td>
</tr>
<tr>
<td>1.96875</td>
</tr>
<tr>
<td>1.984375</td>
</tr>
<tr>
<td>1.9921875</td>
</tr>
<tr>
<td>1.99609375</td>
</tr>
<tr>
<td>1.998046875</td>
</tr>
</tbody>
</table>

1
0.5
0.25
0.125
0.0625
0.03125
0.015625
0.0078125
0.00390625
0.001953125
Definition 4 A set $\mathcal{E} \subset \mathbb{R}_+$ is called an *admissible event times set*, if it is closed and countable, and $0 \in \mathcal{E}$. E.g. $\mathcal{E} = \{\tau_0, \tau_1, \tau_2, \ldots\}$.

- An element $t$ of a set $\mathcal{E}$ is said to be a **left accumulation point** of $\mathcal{E}$, if for all $t' > t$ $(t, t') \cap \mathcal{E}$ is not empty.
- It is called a **right accumulation point**, if for all $t' < t$ $(t', t) \cap \mathcal{E}$ is not empty.

Definition 5 An admissible event times set $\mathcal{E}$ (or the corresponding solution) is said to be **left (right) Zeno free**, if it does not contain any left (right) accumulation points.

- Bouncing ball $\rightarrow$ right accumulation point ... 

- Time-reversed bouncing ball:
Two-tank system and Zeno behavior
A simulation

\[ h_1 = h_2 = 1, \quad q_1 = 2, \quad q_2 = 3, \quad q_{in} = 4, \quad x_1(0) = x_2(0) = 2, \quad q(0) = v_1 \]
Two-tank system and Zeno behavior

- Assume total outflow $q_1 + q_2 > q_{in}$
- Control objective cannot be met and tanks will be empty in finite time
- Infinitely many switchings in finite time (right accumulation point) $\rightarrow$ right Zeno behavior

Using a non-Zeno solution concept: analysis will show that tanks do not get empty! Analysis depends crucially on solution concept!
Hybrid automaton

Hybrid automaton $H$ is collection $H = (Q, X, f, \text{Init}, \text{Inv}, E, G, R)$ with

- $Q = \{q_1, \ldots, q_N\}$ is finite set of discrete states or modes
- $X = \mathbb{R}^n$ is set of continuous states
- $f : Q \times X \to X$ is vector field
- $\text{Init} \subseteq Q \times X$ is set of initial states
- $\text{Inv} : Q \to P(X)$ describes the invariants
- $E \subseteq Q \times Q$ is set of edges or transitions
- $G : E \to P(X)$ is guard condition
- $R : E \to P(X \times X)$ is reset map
What is what?

Hybrid automaton $H = (Q, X, f, \text{Init}, \text{Inv}, E, G, R)$

- Hybrid state: $(q, x)$
- Evolution of continuous state in mode $q$: $\dot{x} = f(q, x)$
- Invariant $\text{Inv}$: describes conditions that continuous state has to satisfy at given mode
- Guard $G$: specifies subset of state space where certain transition is enabled
- Reset map $R$: specifies how new continuous states are related to previous continuous states
\begin{align*}
(q_0, x_0) \in \text{Init} \\
q_0 & \quad \dot{x} = f(q_0, x) \\
& \quad x \in \text{Inv}(q_0) \\
q_1 & \quad \dot{x} = f(q_1, x) \\
& \quad x \in \text{Inv}(q_1) \\
q_2 & \quad \dot{x} = f(q_2, x) \\
& \quad x \in \text{Inv}(q_2)
\end{align*}
Evolution of hybrid automaton

- Initial hybrid state \((q_0, x_0) \in \text{Init}\)
- Continuous state \(x\) evolves according to \(\dot{x} = f(q_0, x)\) with \(x(0) = x_0\)
- Discrete state \(q\) remains constant: \(q(t) = q_0\)
- Continuous evolution can go on as long as \(x \in \text{Inv}(q_0)\)
- If at some point state \(x\) reaches guard \(G(q_0, q_1)\), then
  - transition \(q_0 \rightarrow q_1\) is enabled
  - discrete state may change to \(q_1\), continuous state then jumps from current value \(x^-\) to new value \(x^+\) with \((x^-, x^+) \in R(q_0, q_1)\)
- Next, continuous evolution resumes and whole process is repeated
Hybrid time trajectory

Definition 6 A hybrid time trajectory \( \tau = \{I_i\}_{i=0}^N \) is a finite \((N < \infty)\) or infinite \((N = \infty)\) sequence of intervals of the real line, such that

- \( I_i = [\tau_i, \tau_i'] \) with \( \tau_i \leq \tau_i' = \tau_{i+1} \) for \( 0 \leq i < N \);
- if \( N < \infty \), either \( I_N = [\tau_N, \tau_N'] \) or \( I_N = [\tau_N, \tau_N) \) with \( \tau_N \leq \tau_N' \leq \infty \).

- For instance,
  \[
  \tau = \{[0, 2], [2, 3], \{3\}, \{3\}, [3, 4.5], \{4.5\}, [4.5, 6]\}
  \]
  \[
  \tau = \{[0, 2], [2, 3], [3, 4.5], \{4.5\}, [4.5, 6], [6, \infty)\}
  \]
  \[
  I_i = [1 - 2^i, 1 - 2^{i+1}]
  \]
- \( \mathcal{E} = \{\tau_0, \tau_1, \tau_2, \ldots\} \)
- No left-accumulations of event times ...
Execution of hybrid automaton

**Definition 7** An execution $\chi$ of a HA consists of $\chi = (\tau, q, x)$

- $\tau$ a hybrid time trajectory;
- $q = \{q_i\}_{i=0}^{N}$ with $q_i : I_i \rightarrow Q$; and
- $x = \{x_i\}_{i=0}^{N}$ with $x_i : I_i \rightarrow X$

**Initial condition** $(q(\tau_0), x(\tau_0)) \in \text{Init};$

**Continuous evolution** for all $i$

- $q_i$ is constant, i.e. $q_i(t) = q_i(\tau_i)$ for all $t \in I_i$;
- $x_i$ is solution to $\dot{x}(t) = f(q_i(t), x(t))$ on $I_i$ with initial condition $x_i(\tau_i)$ at $\tau_i$;
- for all $t \in [\tau_i, \tau_i']$ it holds that $x_i(t) \in \text{Inv}(q_i(t))$.

**Discrete evolution** for all $i$,

- $e = (q_i(\tau'_i), q_{i+1}(\tau'_{i+1})) \in E,$
- $x(\tau'_i) \in G(e);$  
- $(x_i(\tau'_i), x_{i+1}(\tau_{i+1})) \in R(e)$.  

Well-posedness for hybrid automata

- $\mathcal{H}_\infty^{(q_0,x_0)}$: infinite executions: $\tau$ is an infinite sequence or if $\sum_i (\tau'_i - \tau_i) = \infty$

- $\mathcal{H}_M^{M^{(q_0,x_0)}}$: maximal executions: $\tau$ is not a strict prefix of another one!

- A hybrid automaton is called non-blocking, if $\mathcal{H}_\infty^{(q_0,x_0)}$ is non-empty for all $(q_0, x_0) \in \text{Init}$.

- It is called deterministic, if $\mathcal{H}_M^{(q_0,x_0)}$ contains at most one element for all $(q_0, x_0) \in \text{Init}$.
Well-posedness for hybrid automata - continued

Assumption

- The vector field $f(q, \cdot)$ is globally Lipschitz continuous for all $q \in Q$.
- The edge $e = (q, q')$ is contained in $E$ if and only if $G(e) \neq \emptyset$ and $x \in G(e)$ if and only if there is an $x' \in X$ such that $(x, x') \in R(e)$.

A state $(\hat{q}, \hat{x}) \in \text{Reach}$, if there exists a finite execution $(\tau, q, x)$ with $\tau = \{[\tau_i, \tau'_i]\}_{i=0}^{N}$ and $(q(\tau'_N), x(\tau'_N)) = (\hat{q}, \hat{x})$.

The set of states from which continuous evolution is impossible:

$$
\text{Out} = \{(q_0, x_0) \in Q \times X \mid \forall \varepsilon > 0 \exists t \in [0, \varepsilon) \ x_{q_0,x_0}(t) \not\in \text{Inv}(q_0)\}
$$

in which $x_{q_0,x_0}(\cdot)$ denotes the unique solution to $\dot{x} = f(q_0, x)$ with $x(0) = x_0$. 
Well-posedness theorems

**Theorem** A hybrid automaton is non-blocking, if for all \((q, x) \in \text{Reach} \cap \text{Out}\), there exists \(e = (q, q') \in E\) with \(x \in G(e)\). In case the automaton is deterministic, this condition is also necessary.

**Theorem** A hybrid automaton is deterministic, if and only if for all \((q, x) \in \text{Reach}\)

- if \(x \in G((q, q'))\) for some \((q, q') \in E\), then \((q, x) \in \text{Out}\);
- if \((q, q') \in E\) and \((q, q'') \in E\) with \(q' \neq q''\), then \(x \not\in G((q, q')) \cap G((q, q''))\); and
- if \((q, q') \in E\) and \(x \in G((q, q'))\), then there is at most one \(x' \in X\) with \((x, x') \in R((q, q'))\).

→ no explicit / algebraic conditions and not easily verifiable → can we do more (like for DDE)?
Well-posedness issues

- **Initial well-posedness**: non-blocking + deterministic, i.e. absence of
  - **dead-lock**: no smooth continuation and no jump
  - splitting of trajectories

However, no statements by HA theory on existence beyond

- **live-lock**: an infinite number of jumps at one time instant, no solution on $[0, \varepsilon)$ for some $\varepsilon > 0$.

- **right-accumulations** of event times to prevent global existence.

or absence of

- **left-accumulations** of event times preventing uniqueness:
Obstruction local existence

→ **Live-lock**: Infinitely many jumps at one time instant

V₁(0)=1     V₂(0)=0     V₃(0)=0

Ball 1     Ball 2     Ball 3

\[ v_1 : \quad 1 \quad {\frac{1}{2}} \quad {\frac{1}{2}} \quad {\frac{3}{8}} \quad {\frac{3}{8}} \quad {\frac{11}{32}} \quad \ldots \quad {\frac{1}{3}} \]

\[ v_2 : \quad 0 \quad {\frac{1}{2}} \quad {\frac{1}{4}} \quad {\frac{3}{8}} \quad {\frac{5}{16}} \quad {\frac{11}{32}} \quad \ldots \quad {\frac{1}{3}} \]

\[ v_3 : \quad 0 \quad 0 \quad {\frac{1}{4}} \quad {\frac{1}{4}} \quad {\frac{5}{16}} \quad {\frac{5}{16}} \quad \ldots \quad {\frac{1}{3}} \]

- smooth continuation possible with constant velocity after an infinite number of events

→ Exclude live-lock or show convergence of state \( x \) for local existence

- Discrete mode is a function of continuous state! not for general HA!!!
Obstruction global existence: Zenoness

→ A right-accumulation of event times

\[
\begin{align*}
\dot{x}_1 &= -\text{sgn}(x_1) + 2 \text{sgn}(x_2) \\
\dot{x}_2 &= -2 \text{sgn}(x_1) - \text{sgn}(x_2)
\end{align*}
\]

- Exclude right-accumulations or show the existence of the left-limit
  \( \lim_{t \uparrow \tau^*} x(t) \) for global existence.
- **Discrete mode is a function of continuous state!** not for general HA!!!
Obstructions local uniqueness: Filippov’s example

\begin{align*}
\dot{x}_1 &= \text{sgn}(x_1) - 2\text{sgn}(x_2) \\
\dot{x}_2 &= 2\text{sgn}(x_1) + \text{sgn}(x_2),
\end{align*}

Left accumulation point ... \( \mathcal{E} \) is not left Zeno free!

Well-posedness:

- Due to left-accumulations non-uniqueness in origin
- Using HA framework: non-blocking and deterministic
- Using Filippov’s solution: non-uniqueness!
Well-posedness

- **Initially solvable** from each initial state there exists a state jump or a continuous hybrid solution on \([0, \varepsilon)\) (non-blocking)
- **Initially unique** from each initial state the jump/hybrid solution is unique (deterministic)
- **Local well-posedness** from each initial state there exists an \(\varepsilon > 0\) and a hybrid solution on \([0, \varepsilon)\).
- **Global well-posedness** ... on \([0, \infty)\).
Piecewise linear systems

\[ \text{SAT}(A, B, C, D) \]
\[ \dot{x}(t) = Ax(t) + Bu(t) \quad e^i_2 - e^i_1 > 0 \text{ and } f^i_1 \geq f^i_2 \]
\[ y(t) = Cx(t) + Du(t) \]
\[ (u(t), y(t)) \in \text{saturation}_i \]

Note that if \( f^i_2 = f^i_1 \), then relay-type of nonlinearity.
Example of linear relay system: non-uniqueness

\[
\dot{x} = x - u \\
y = x \\
u \in -\text{sgn}(y)
\]

\[
x(0) = 0:
\]

- \(x(t) = e^t - 1, (y(t) = x(t) \geq 0)\)
- \(x(t) = -e^t + 1, (y(t) = x(t) \leq 0)\)
- \(x(t) = 0, (y(t) = x(t) = 0)\)
Example of linear relay system: uniqueness

\[ \dot{x} = x + u \]
\[ y = x \]
\[ u \in -\text{sgn}(y) \]

\[ x(0) = 0: \]

- \( x(t) = 0, (y(t) = x(t) = 0) \)
Consider \( \text{SAT}(A, B, C, D) \).

- Let \( R \) and \( S \) be the diagonal matrices with \( e_i^2 - e_i^1 \) and \( f_i^2 - f_i^1 \), resp.
- \( G(s) = C(sI - A)^{-1}B + D \)

Suppose that \( G(\sigma)R - S \) is a \( P \)-matrix for all sufficiently large \( \sigma \). Then, there exists a unique (left Zeno free) hybrid execution of \( \text{SAT}(A, B, C, D) \) for all initial states.

- \( M \in \mathbb{R}^{m \times m} \) is a \( P \)-matrix, if \( \det M_I > 0 \) for all \( I \subseteq \{1, \ldots, m\} \).
Linear relay systems and Filippov’s solution concept: left accumulations

\[ \dot{x}(t) = Ax(t) + Bu(t); \quad y(t) = Cx(t); \quad u(t) \in -\text{sgn}(y(t)) \]

Previous result: If \( G(\sigma) = CB\sigma^{-1} + CAB\sigma^{-2} + \ldots > 0 \) for sufficiently large \( \sigma \), then existence and uniqueness of (left-Zeno free) executions.
Other solution concept ...?

Filippov’s solutions include left-accumulations and satisfy $\dot{x} \in F(x)$ almost everywhere, with

- $F(x) = \{Ax + B\}$ for $Cx < 0$
- $F(x) = \{Ax - B\}$ for $Cx > 0$
- $F(x) = \{Ax + B\bar{u} | \bar{u} \in [-1, 1]\}$ when $Cx = 0$

In case of relative degree 1 ($CB > 0$) and relative degree 2 (and order 2) sufficient for Filippov uniqueness.

However, triple integrator $\frac{d^3 x}{dt^3} = u$ counterexample due to:

So, (other) example of HA uniqueness (deterministic), but non-uniqueness in “Filippov”
Linear complementarity systems

\[ \dot{x}(t) = Ax(t) + Bz(t) \]
\[ w(t) = Cx(t) + Dz(t) \]
\[ 0 \leq w(t) \perp z(t) \geq 0 \]

\[ \{ z_i(t) = 0 \text{ and } w_i(t) \geq 0 \} \text{ or } \{ w_i(t) = 0 \text{ and } z_i(t) \geq 0 \} \]

- modes parameterized by \( I \subseteq \{1, \ldots, k\} \) such that

\[ \dot{x}(t) = Ax(t) + Bz(t) \]
\[ w(t) = Cx(t) + Dz(t) \]
\[ w_i = 0, \ i \in I \ \text{ and } \ z_i = 0, \ i \notin I \]
Example 1

\[ \dot{x} = x + z \]
\[ w = x - z \]
\[ 0 \leq w \perp z \geq 0 \]

- \( z = 0 \): \( \dot{x} = x, w = x \geq 0 \)
- \( w = 0 \): \( \dot{x} = 2x, z = x \geq 0 \)

Hence, \( x(0) = 1 \) two solutions and \( x(0) = -1 \) no solution trajectory!
Example 2

\[
\begin{align*}
\dot{x} &= x + z \\
w &= x + z \\
0 &\leq w \perp z \geq 0
\end{align*}
\]

- \( z = 0 \): \( \dot{x} = x, w = x \geq 0 \)
- \( w = 0 \): \( \dot{x} = 0, z = -x \geq 0 \)

Existence and uniqueness!

Model test ...
Well-posedness including jumps

- **Initially solvable** from each initial state there exists a state jump or a continuous hybrid solution on $[0, \varepsilon)$ (non-blocking)

- **Initially unique** from each initial state the jump/hybrid solution is unique (deterministic)

- **Local well-posedness** from each initial state there exists an $\varepsilon > 0$ and a hybrid solution on $[0, \varepsilon)$.

- **Global well-posedness** ... on $[0, \infty)$.
Local well-posedness (including jumps)

\[ \dot{x}(t) = Ax(t) + Bz(t), \quad w(t) = Cx(t) + Dz(t), \quad 0 \leq z(t) \perp w(t) \geq 0 \]

Markov parameters: \( H^0 = D \) and \( H^i = CA^{i-1}B, \ i = 1, 2, \ldots \)

\[ \eta_j = \inf\{i \mid H^i_{\cdot j} \neq 0\}, \ \rho_j = \inf\{i \mid H^i_{j \cdot} \neq 0\}, \]

The leading row and column coefficient matrices \( \mathcal{M} \) and \( \mathcal{N} \)

\[
\mathcal{M} := \begin{pmatrix}
H_{1 \cdot}^{\rho_1} \\
\vdots \\
H_{k \cdot}^{\rho_k}
\end{pmatrix}
\quad \text{and} \quad
\mathcal{N} := (H_{\cdot 1}^{\eta_1} \ldots H_{\cdot k}^{\eta_k})
\]

- \( M \in \mathbb{R}^{m \times m} \) is a \textit{P-matrix}, if \( \det M_{II} > 0 \) for all \( I \subseteq \{1, \ldots, m\} \).

If \( \mathcal{N} \) and \( \mathcal{M} \) are defined and P-matrices, then \( \text{LCS}(A, B, C, D) \) has for all \( x_0 \) a unique left Zeno free execution on an interval of the form \( [0, \varepsilon) \) for some \( \varepsilon > 0 \).

- Moreover, live-lock does not occur: at most one jump
- Necessary and sufficient for \textit{global} well-posedness for \textit{bimodal} LCS
Summary

• Smooth differential equations
  – Solution concept straightforward
  – Lipschitz continuity sufficient for well-posedness
  – absence Lipschitz: possibly non-uniqueness
  – absence global Lipschitz finite escape times and no global existence

• Switched systems (discontinuous differential equations)
  – Sliding modes (Filippov’s convex or Utkin’s equivalent control definition)
  – Solution concept from differential inclusions
  – Well-posedness: directions of vector field at switching plane

“No events”
Summary - continued

- Hybrid systems:
  - Complications due to Zeno
  - Relation between solution concept and well-posedness and analysis
    * Tanks stay full along non-Zeno solutions!!!
    * Filippov’s example has unique non-Zeno solutions, but non-unique Zeno solutions
  - Well-posedness
    * Initial well-posedness (non-blocking and deterministic)
    * Local well-posedness: \([0, \varepsilon)\) (live-lock)
    * Global well-posedness: \([0, \infty)\) (right-accumulations)
  - Conditions for hybrid automata: implicit!
  - Algebraic conditions for certain classes with more structure!
Selected Literature


