The dual of the maximum flow problem

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Given a network $G = (N, A)$, and two nodes s (source) and t (sink), the maximum flow problem can be formulated as:

$$\max \quad v \tag{1}$$

$$\sum_{(s,j)\in\delta^+(s)} x_{sj} = v \tag{2}$$

$$-\sum_{(i,t)\in\delta^-(t)} x_{it} = -v \tag{3}$$

$$\sum_{(h,j)\in\delta^+(h)} x_{hj} - \sum_{(i,h)\in\delta^-(h)} x_{ih} = 0, \quad h \in N - \{s,t\} \tag{4}$$

$$x_{ij} \leq k_{ij} \quad (i,j) \in A \tag{5}$$

$$x_{ij} \geq 0 \quad (i,j) \in A \tag{6}$$

where variables x_{ij} indicate the flow in arc (i,j) and v is the value of the flow to be maximized. Let us now associate dual variables u_s, u_t and u_j $(j \neq s,t)$ to constraints (2),(3) and (4) respectively, and variables y_{ij}, $(i,j) \in A$ to constraints (5), one can write the dual problem:

$$\min \sum_{(i,j)\in A} k_{ij} y_{ij} \tag{7}$$

$$u_i - u_j + y_{ij} \geq 0, (i,j) \in A \tag{8}$$

$$-u_s + u_t = 1 \tag{9}$$

$$y_{ij} \geq 0 \tag{10}$$

Let us analyze the structure of basic feasible solutions of the dual problem. First we make two simple observations. The first is that the matrix of the coefficients of the dual problem is the transpose of the matrix in the corresponding primal, so it is totally unimodular. Hence, we are sure that all basic solutions will have integer components (assuming of course integer values k_{ij}). The second observation is that, in the dual

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problem, the variables $\{u_j\}$ do not appear in the objective function, and in each constraint the *difference* of two of these variables appears. So, in the dual problem, what is relevant for the purpose of minimization is *not* the absolute value of these variables, but their relative value. In other words, given any feasible dual solution $\{u_j, y_{ij}\}$, if one adds the same constant value to all variables $\{u_j\}$, another feasible solution is obtained, having the same value of the objective function. Ultimately, one can arbitrarily fix the value of one of these variables. In particular, fix $u_s = 0$. In view of constraint (9), we can therefore limit ourselves to considering basic solutions in which $u_s = 0$ and $u_t = 1$, and in which all variables have integer values.

TEOREMA 1 *Given a cut $[S, \bar{S}]$ on G, it is possible to associate to it a feasible solution of the dual defined as:*

- $u_i = 0$ *if* $i \in S$
- $u_i = 1$ *if* $i \notin S$
- $y_{ij} = 1$ *if* $(i, j) \in (S, \bar{S})$
- $y_{ij} = 0$ *if* $(i, j) \notin (S, \bar{S})$

Proof.– To prove the result, it suffices to show that the constraints of the dual problem are all satisfied by the solution. Given the constraint of the dual corresponding to the arc $(i, j) \in A$, we consider the four possible cases.

(i) $i \in S, j \in S$. In this case, both endpoints of arc (i, j) belong to S: in the corresponding constraint (8) one has $u_i = 0, u_j = 0$ and also $y_{ij} = 0$ (since the arc is entirely in S), and hence the constraint is active.

(ii) $i \notin S, j \notin S$. In this case, both endpoints of arc (i, j) *do not* belong to S, and in (8) one has $u_i = 1, u_j = 1$ and $y_{ij} = 0$, and hence also in this case the constraint is active.

(iii) $i \in S, j \notin S$. In this case, arc (i, j) is an outgoing arc of S, and so $y_{ij} = 1$. Moreover $u_i = 0$ and $u_j = 1$, and hence again the constraint is active.

(iv) $i \notin S, j \in S$. In this case, arc (i, j) is an ingoing arc of S, and so $y_{ij} = 0$, while $u_i = 1$ and $u_j = 0$. The constraint (8) is strictly satisfied.

□

The previous theorem states that we can associate a feasible solution of the dual to each cut $[S, \bar{S}]$. Note that, since the only variables $\{y_{ij}\}$ equal to 1 are those corresponding to $(S, \bar{S})$, the value of the feasible solution is equal to the capacity of the cut $[S, \bar{S}]$. In what follows, given a cut $[S, \bar{S}]$, we say that the solution defined in Theorem 1 is the *cut solution* $u(S), y(S)$.

Interestingly, the converse result of Theorem 1 also holds.

TEOREMA 2 *Given a feasible solution $\{u_j, y_{ij}\}$ of the dual (in which $u_s = 0$ and $u_t = 1$) of value w, it is possible to find a cut $[S, \bar{S}]$ such that the value of the corresponding cut solution $u(S), y(S)$ does not exceed w.*

Dim. Consider any dual solution $\{u_j, y_{ij}\}$. Note that, for each arc (i, j), the variable y_{ij} only appears in *one* constraint (8), and hence we can suppose, with no loss of generality, that the value of y_{ij} equals the minimum that allows to satisfy the corresponding constraint, i.e.

$$y_{ij} = \max\{0, u_j - u_i\}. \tag{11}$$

In fact, higher values of y_{ij} would only increase the objective function. Moreover, due to the total unimodularity of the coefficient matrix, we can assume that all dual variables are integer. Since $u_s = 0$ and $u_t = 1$, *along each path from s to t* one will always meet at least one arc (i, j) such that $y_{ij} > 0$, i.e., $y_{ij} \geq 1$. This occurs because from the value 0 it will be necessary sooner or later to "climb" at least to the value 1, and so at least one variable y_{ij} must have a positive value.

Now remove from the graph all the arcs such that $y_{ij} > 0$, and let S be the set of nodes reachable from s (clearly t is no more reachable from s.) Let $A(S)$ and $A(\bar{S})$ be the sets of arcs having both endpoints in S and, respectively, in $\bar{S}$. Clearly, all nodes in S have $u_j = 0$, since otherwise (11) would imply $y_{ij} > 0$ for some arc $(h, j) \in A(S)$. Analogously, for all nodes in $\bar{S}$ we can set $u_j = 1$, since in this way all variables y_{ij} corresponding to arcs of $A(\bar{S})$ can be set to zero. At this point, for all arcs (i, j) with $i \notin S$ and $j \in S$, (11) yields $y_{ij} = 0$. In conclusion, we can reduce to a solution in which the only nonzero variables y_{ij} are equal to 1, and are those corresponding to the arcs of $(S, \bar{S})$, i.e., to forward arcs of the cut $[S, \bar{S}]$ (these are the arcs along which the node potential can climb from 0 to 1, when following a path from s to t). This is precisely the cut solution $y(S), u(S)$, whose value coincides with the capacity of the cut $[S, \bar{S}]$. $\square$

This theorem therefore shows that the dual of the maximum flow problem is the problem of finding a cut of minimum capacity, and that therefore the well-known max-flow/min-cut theorem is simply a special case of the strong duality theorem. However, the solution approach by Ford and Fulkerson also provides an efficient solution algorithm.

Applying complementary slackness conditions to such primal-dual pair gives some interesting insight. From (5), if, in the optimal solution of the dual problem, $y_{ij} = 1$, then $x^*_{ij} = k_{ij}$, i.e., the corresponding arc must be saturated. In fact, $y_{ij} = 1$ only if $(i, j) \in (S^*, \bar{S}^*)$, where $[S^*, \bar{S}^*]$ is a minimum cut. Moreover, consider a constraint (8) which is not active in the optimal solution of the dual, i.e., $u^*_i - u^*_j + y^*_{ij} > 0$. In this case, the corresponding primal variable x^*_{ij} must be zero, which is correct since the only case where the constraint (8) is not active corresponds to a backward arc of the cut $[S^*, \bar{S}^*]$, which must therefore be empty.