Analysis of consensus protocols with bounded measurement errors

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Abstract

This paper analyzes two classes of consensus algorithms in presence of bounded measurement errors. The considered protocols adopt an updating rule based either on constant or vanishing weights. Under the assumption of bounded error, the consensus problem is cast in a set-membership framework, and the agreement of the team is studied by analyzing the evolution of the feasible state set. Bounds on the asymptotic difference between the states of the agents are explicitly derived, in terms of the bounds on the measurement noise and the values of the weight matrix.

Key words: Consensus protocols, Set-membership uncertainty, Multi-agent systems

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1. Introduction

In recent years consensus algorithms have received increasing interest within the context of multi-agent systems. The ability of a team of interacting agents to reach an agreement on some quantity of interest is often a key issue for the solution of many problems in different application domains, like distributed sensing [1], cooperative control of autonomous vehicles [2], rendezvous algorithms for autonomous agents [3], coordination of robotic networks [4]. The reader is referred to [5] for an exhaustive presentation of possible applications of consensus algorithms. A number of solutions to the consensus problem have been proposed by now, and nice theoretical results are available for both stationary and time-varying communication networks (e.g., see [6, 7, 8] and the survey [9])

Compared to the large amount of papers analyzing how the topology of the communication graph affects the convergence properties of the consensus protocols, relatively few ones have addressed the behavior of consensus algorithms in presence of noisy measurements. In [10] classical consensus algorithms are shown to be input-to-state stable. This property is exploited in [11] to devise a consensus algorithm for tracking the average of time-varying signals. A consensus protocol with vanishing weights, mimicking stochastic approximation algorithms with a decreasing step size, has been proposed in [12]. The authors show that in

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case of measurements affected by stochastic noise, the adoption of vanishing weights guarantees the convergence in probability of the agents' states to the same value. A different description of the uncertainty is adopted in [13] and [14], where the measurement noise is only assumed to be bounded. In [13], the authors propose a nonlinear consensus protocol that ensures the convergence of the states in a tube whose radius depends on the maximum amplitude of the measurement noise. In [14] a consensus-based rendezvous algorithm, ensuring the convergence of the system towards a ball with a finite radius in presence of bounded noise, is presented.

In this paper we analyze two classes of consensus algorithms in a set-theoretic framework. Under the assumption of unknown but bounded measurement errors, the feasible state set (i.e., the set of all states compatible with the bounds on the noise) is explicitly derived. This kind of sets naturally arise in the context of set-membership estimation theory, which was originally developed for dynamic system identification and filtering problems, to guarantee a worst-case bound in the estimation of the model parameters or of the state vector [15, 16]. The evolution of the feasible state set is used to evaluate the agreement of the team. Linear consensus protocols adopting both constant weights and vanishing weights are considered, in the case of undirected and stationary communication graph. It is shown that for both types of protocols, asymptotic consensus cannot be guaranteed with respect to all possible noise realizations, and bounds on the asymptotic difference of the agents' states are explicitly derived, as a function of the bounds on the measurement errors and the weight matrix. This work extends previous results presented in [17], where the weight matrix was supposed to be symmetric.

The paper is organized as follows. Section 2 provides an overview of the consensus protocols to be analyzed. In Section 3 the consensus problem is formulated in a set-membership framework, under the assumption of bounded measurement errors. The main contributions of the paper are presented in Section 4, where the asymptotic difference among the agents' states is related to the bounds on the measurement noise and to the weights used in the consensus protocol. A numerical example, illustrating the obtained results, is reported in Section 5. Finally, in Section 6 some conclusions are drawn and future directions of research are outlined.

1.1. Notation

The ith component of a vector $x \in \mathbb{R}^n$ is denoted by x_i. The symbol $\mathbf{1}$ denotes the vector whose components are all equal to 1, $\mathbf{1} = [1 \ \dots \ 1]'$. Given a real matrix $A \in \mathbb{R}^{n \times n}$, A_i is the ith column of A and A_{ij} is the ijth element of A. The eigenvalues of A are denoted by $\lambda_i(A)$, and if A is symmetric they are conventionally labeled in decreasing order, $\lambda_1(A) \geq \cdots \geq \lambda_n(A)$ (unless otherwise specified). The singular value decomposition (SVD) of a matrix A is denoted by $Y^A \Sigma^A U^{A'}$, where Σ^A is the diagonal matrix containing the singular values $\sigma_i(A)$. The singular values of A are conventionally labeled in decreasing order, $\sigma_1(A) \geq \cdots \geq \sigma_n(A)$ (unless otherwise specified). The 2-norm of A is defined as $\|A\|_2 = \sigma_1(A)$. A non-negative matrix A is a matrix whose entries are all non-negative. An undirected graph $\mathcal{G}$ is a pair $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ denotes the vertex set and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the edge set. An edge (i, j), $i \neq j$, belongs to $\mathcal{E}$ if and only agents i and j can communicate.

Since $\mathcal{G}$ is undirected, if $(i,j) \in \mathcal{E}$ then also $(j,i) \in \mathcal{E}$. A path between two vertices $i,j \in \mathcal{V}$ is a sequence of edges $(l_k, l_{k+1}) \in \mathcal{E}$, $k = 1,\ldots,s-1$ such that $l_1 = i$ and $l_s = j$. The graph $\mathcal{G}$ is connected if there exists a path between any two nodes $i,j \in \mathcal{V}$. Finally, we denote by $\mathcal{N}_i$ the set of neighbors of agent i, i.e. $\mathcal{N}_i = \{j \in \mathcal{V} : (i,j) \in \mathcal{E}\}$.

2. Motivation and related work

Consider a system of n agents $\mathcal{V} = \{1,\ldots,n\}$ communicating among them according to an undirected graph $\mathcal{G}$. Let $x_i(t) \in \mathbb{R}$ be the state of agent i at time $t \in \mathbb{N}$. At the same time instant, agent i is given a noisy measurement of the state of all its neighbors

$$y_{ij}(t) = x_j(t) + \eta_{ij}(t), \quad i = 1,\ldots,n, \quad j \in \mathcal{N}_i. \tag{1}$$

The term $\eta_{ij}(t)$ models the uncertainty affecting the knowledge of the state of agent j, from agent i's viewpoint.

Each agent updates its state according to the equation

$$x_i(t+1) = x_i(t) + u_i(t), \quad i = 1,\ldots,n,$$

where $u_i(t)$ is the input of the i-th agent. If we denote by $Y^{(i)}(t) = \{y_{ij}(t)\}_{j\in\mathcal{N}_i}$ all the information available to agent i at time t, the objective of a consensus algorithm is to find for each agent a control law $u_i(t) = f\left(x_i(t), Y^{(i)}(t)\right)$, $i = 1,\ldots,n$, ensuring the convergence of the agents' state to a common value, i.e. such that

$$\lim_{t\to\infty} |x_i(t) - x_j(t)| = 0, \quad i,j = 1,\ldots,n.$$

In the ideal case of noiseless information (i.e., when $\eta_{ij}(t) = 0, \ \forall t$), a number of different solutions have been proposed, both for stationary and time-varying topology of the communication network, as well as for directed and undirected communication graphs (see, e.g., [5, 9]). The vast majority of the proposed algorithms adopt a feedback control law $f(\cdot)$ which is a linear function of the agent states, the so-called *linear consensus protocols*. When the topology of the communication graph is stationary, a linear consensus protocol takes on the form

$$x_i(t+1) = x_i(t) + \sum_{j\in\mathcal{N}_i} w_{ij}(x_j(t) - x_i(t)), \quad i = 1,\ldots,n, \tag{2}$$

where the constants w_{ij} are the *weights* of the consensus protocol. If we stack the states of all the agents into a single vector $x(t) = [x_1(t) \ \ldots \ x_n(t)]'$, equations (2) can be rewritten in vector form as

$$x(t+1) = (I+W)x(t).$$

Clearly, the communication graph determines the sparsity pattern of W. It is well-known that if the graph is connected then there exist many possible choices of the weight matrix ensuring consensus, and, if in addition W is symmetric, the consensus value is simply the average of the initial agents' states (*average consensus* problem, [18]).

When the true state is not accessible, and noisy measurements like in (1) are used to replace the actual state value, the updating rule (2) becomes

$$x_i(t+1) = x_i(t) + \sum_{j\in\mathcal{N}_i} w_{ij}(y_{ij}(t) - x_i(t)) = x_i(t) + \sum_{j\in\mathcal{N}_i} w_{ij}(x_j(t) - x_i(t)) + \sum_{j\in\mathcal{N}_i} w_{ij}\eta_{ij}(t) \quad (3)$$

for $i = 1, \ldots, n$. Let $v(t) = [v_1(t), \ \ldots, \ v_n(t)]'$, where

$$v_i(t) = \sum_{j\in\mathcal{N}_i} w_{ij}\eta_{ij}(t), \quad i = 1, \ldots, n, \quad (4)$$

then equations (3) can be rewritten in vector form as

$$x(t+1) = (I + W)x(t) + v(t). \quad (5)$$

Due to the presence of the forcing term $v(t)$, consensus cannot be guaranteed anymore. However, using input-to-state stability arguments, it can be shown that the maximum difference between any two states remains bounded provided that the measurement noise $\eta_{ij}(t)$ is bounded, and asymptotically vanishes if $\eta_{ij}(t)$ tends to zero [10].

On the other hand, in order to make the effect of a persistent noise tend to zero, the measurements $y_{ij}(t)$ should be weighted lesser and lesser over time. Let $a(t)$ be a positive function such that $\lim_{t\to\infty} a(t) = 0$. If the weights w_{ij} in (3) are replaced by $a(t)w_{ij}$, the updating rule becomes

$$x(t+1) = (I + a(t)W)x(t) + a(t)v(t). \quad (6)$$

In this case the agents' states evolve according to a time-varying linear system, fed by a vanishing input. If the measurement disturbances are modeled as independent stochastic variables, with zero mean and finite variance, then choosing $a(t) \sim \frac{1}{t^r}$, $0.5 < r \leq 1$, ensures that the states of all the agents converge in probability to the same limit [12].

Driven by the aforementioned observations, the objective of this paper is twofold. Suppose W is selected so as to guarantee consensus in the noise-free case. Moreover, assume that the measurement noise $\eta_{ij}(t)$ is bounded. The first goal is to quantify the difference between the states as a function of the noise bound, in case constant weights are used (equation (5)). The second goal is to study the achievement of consensus when vanishing weights are adopted (equation (6)). The bounded error assumption naturally leads to cast these problems in a set-membership framework, as it will be shown in the next section.

3. Set-theoretic consensus

Let the state $x(t)$ be updated according to equation (5), where the input $v(t)$ is given by (4). Assume that the measurement noise is unknown-but-bounded (UBB), i.e.

$$|\eta_{ij}(t)| \leq \bar{\epsilon}, \quad i = 1, \ldots, n, \quad j \in \mathcal{N}_i, \ \forall t, \quad (7)$$

where $\bar{\epsilon} > 0$ is a known quantity. According to (4), the UBB assumption on the measurement noise immediately reflects on the possible values taken by the disturbance $v(t)$, i.e. $|v_i(t)| \leq \epsilon_i$, where $\epsilon_i = \bar{\epsilon} \sum_{j \in \mathcal{N}_i} w_{ij}$.

For a given initial condition $x(0)$, it is possible to define the *feasible state set* (FSS) $\mathcal{X}_C(t)$ through the recursion

$$\begin{aligned} \mathcal{X}_C(0) &= \{x(0)\} \\ \mathcal{X}_C(t+1) &= (I+W)\mathcal{X}_C(t) \oplus D_\epsilon \mathcal{B}_\infty, \end{aligned} \tag{8}$$

where $\mathcal{B}_\infty$ denotes the unit ball in the ∞-norm, defined as $\|x\|_\infty = \max_i |x_i|$, and D_ϵ is the diagonal matrix whose i-th entry on the diagonal is equal to ϵ_i. In (8), the symbol $\oplus$ denotes the sum of sets. The set $D_\epsilon \mathcal{B}_\infty$ is a box in $\mathbb{R}^n$ and contains all the possible realizations of the disturbance $v(t)$ which satisfy the UBB assumption (7). Consequently, the set $\mathcal{X}_C(t)$ contains all the states at time t compatible with the initial condition $x(0)$ and the error bounds (7). By expanding the recursion (8), it can be checked that the feasible state set at time t can be written as

$$\mathcal{X}_C(t) = \{x \in \mathbb{R}^n : x = x_c(t) + T(t)\alpha,\ \|\alpha\|_\infty \leq 1\} \tag{9}$$

where

$$\begin{aligned} x_c(t) &= F^t x(0), \\ T(t) &= [T_1\ T_2\ \ldots\ T_t] \in \mathbb{R}^{n \times tn}, \\ F &= I + W, \\ T_i &= F^{t-i} D_\epsilon \in \mathbb{R}^{n \times n}, \quad i = 1, \ldots, t. \end{aligned} \tag{10}$$

The set $\mathcal{X}_C$ is a *parpolygon* in $\mathbb{R}^n$, with center x_c and edges parallel to the columns s_i of $T = [s_1\ \ldots\ s_{tn}]$ (see, e.g., [16]).

Similarly to the case of constant weights, if the state evolves according to equation (6), under the UBB assumption (7), the feasible state set $\mathcal{X}_V(t)$ at time t is given by

$$\begin{aligned} \mathcal{X}_V(0) &= \{x(0)\} \\ \mathcal{X}_V(t+1) &= (I + a(t)W)\mathcal{X}_V(t) \oplus a(t) D_\epsilon \mathcal{B}_\infty. \end{aligned} \tag{11}$$

Hence the feasible state set is still a parpolygon like (9), but with different center and edges

$$\mathcal{X}_V(t) = \{x \in \mathbb{R}^n : x = x_c(t) + T(t)\alpha,\ \|\alpha\|_\infty \leq 1\} \tag{12}$$

where $x_c(t) = \Phi(t,0)x(0)$, $T(t) = [T_1\ T_2\ \ldots\ T_t] \in \mathbb{R}^{n \times tn}$, and

$$\begin{aligned} F(t) &= I + a(t)W, \\ \Phi(t_2, t_1) &= F(t_2 - 1)F(t_2 - 2) \ldots F(t_1), \quad 0 \leq t_1 < t_2, \\ \Phi(t,t) &= I, \\ T_i &= a(i-1)\Phi(t,i) D_\epsilon \in \mathbb{R}^{n \times n}, \quad i = 1, \ldots, t. \end{aligned} \tag{13}$$

Since the states of the agents at time t are constrained to belong to the parpolygon $\mathcal{X}_{(\cdot)}(t)$, the achievement of consensus can be established by studying the time evolution of $\mathcal{X}_{(\cdot)}(t)$. Specifically, consensus is reached if and only if all the segments defining $\mathcal{X}_{(\cdot)}(t)$ (the columns of matrix $T(t)$) eventually align with the vector $\mathbf{1} = [1\ 1\ \dots\ 1]' \in \mathbb{R}^n$, i.e. if and only if the parpolygon degenerates into a line. Moreover, should this not happen, a measure of the disagreement of the team is given by the maximum size of the projection of $\mathcal{X}_{(\cdot)}(t)$ on the subspace orthogonal to $\mathbf{1}$, denoted by $S^{\mathbf{1}^\perp}$. As a matter of fact, let $P^{\mathbf{1}^\perp} = I - \frac{\mathbf{1}\mathbf{1}'}{n}$ be the projection operator of a vector of $\mathbb{R}^n$ on $S^{\mathbf{1}^\perp}$. Then the projection of $\mathcal{X}_{(\cdot)}(t)$ on $S^{\mathbf{1}^\perp}$ is given by $\mathcal{X}^{\mathbf{1}^\perp}_{(\cdot)}(t) = \{x^{\mathbf{1}^\perp} \in \mathbb{R}^n : x^{\mathbf{1}^\perp} = P^{\mathbf{1}^\perp} x,\ x \in \mathcal{X}_{(\cdot)}(t)\}$. and

$$r_{(\cdot)}(t) = \max_{x \in \mathcal{X}^{\mathbf{1}^\perp}_{(\cdot)}(t)} \|x\|_2 \tag{14}$$

is the maximum deviation from consensus (in the 2-norm) at time t. Upper and lower bounds on the asymptotic value of $r_C(t)$ and $r_V(t)$ will be derived in the next section.

4. Bounds on asymptotic disagreement

This section contains the main contributions of the paper. First, we will characterize the feasible state set for the case of constant weights (Section 4.1), and then we will address the same problem in case of vanishing weights (Section 4.2). In both scenarios, we will study two aspects. One concerns the finiteness of the consensus value (Propositions 1 and 2). The other one is related to finding upper and lower bounds on the asymptotic deviation from consensus (Theorems 1-4). The proofs of all the lemmas are reported in the appendix.

Let the weight matrix W be defined as

$$W_{ij} = \begin{cases} w_{ij} > 0 & \text{if } j \in \mathcal{N}_i \\ w_{ii} = -\sum_{j \in \mathcal{N}_i} w_{ij} > -1 & \\ 0 & \text{otherwise} \end{cases}. \tag{15}$$

In the remaining of the paper, the following assumptions are made.

Assumption 1 (UBB noise). *The noise $v(t)$ is bounded: $v(t) \in D_\epsilon \mathcal{B}_\infty$.*

Assumption 2 (Connectivity). *The undirected communication graph $\mathcal{G}$ is connected.*

Assumption 3 (Weights). *The weight matrix satisfies $W\mathbf{1} = W'\mathbf{1} = 0$.*

Assumption 3 is a necessary condition for achieving average consensus [18]. The following identities will be useful for establishing the main results of the paper

$$P^{\mathbf{1}^\perp} W = W P^{\mathbf{1}^\perp} = W$$

$$P^{\mathbf{1}^\perp} F = F P^{\mathbf{1}^\perp} = P^{\mathbf{1}^\perp} F P^{\mathbf{1}^\perp} = F - \frac{\mathbf{1}\mathbf{1}'}{n} = P^{\mathbf{1}^\perp} + W = I - \frac{\mathbf{1}\mathbf{1}'}{n} + W. \tag{16}$$

4.1. Constant weights

Let us suppose that the state of the agents evolves according to the stationary dynamic model (5), which we rewrite as

$$\begin{aligned} x(t+1) &= Fx(t) + v(t) \\ x(0) &= x_0 \end{aligned}, \tag{17}$$

where $F = I + W$ and the weights are chosen as in (15).

Proposition 1. *The feasible state set $\mathcal{X}_C$ given by (9) is asymptotically unbounded along the vector $\mathbf{1}$.*

Proof: Denote by $\mathcal{X}_C^{\mathbf{1}}(t)$ the orthogonal projection of $\mathcal{X}_C(t)$ on the subspace spanned by $\mathbf{1}$, i.e.

$$\mathcal{X}_C^{\mathbf{1}}(t) = \{x^{\mathbf{1}} \in \mathbb{R}^n : x^{\mathbf{1}} = \frac{\mathbf{1}\mathbf{1}'}{n}x,\ x \in \mathcal{X}_C(t)\}.$$

Let us consider the state $\hat{x}(t) \in \mathcal{X}_C(t)$ such that $\hat{x}(t) = x_c(t) + T(t)\hat{\alpha}$, with $\hat{\alpha} = [1\ 1\ ...\ 1]' \in \mathbb{R}^{tn}$. Then, from (9)-(10) and noting that under Assumption 3 $\mathbf{1}'F = \mathbf{1}'$, the projection $\hat{x}^{\mathbf{1}}(t)$ of $\hat{x}(t)$ on $\mathbf{1}$ is given by

$$\begin{aligned} \hat{x}^{\mathbf{1}}(t) &= \frac{\mathbf{1}\mathbf{1}'}{n}\hat{x}(t) = \frac{\mathbf{1}\mathbf{1}'}{n}(x_c(t) + T(t)\hat{\alpha}) = \frac{\mathbf{1}}{n}(\mathbf{1}'x(0) + [\mathbf{1}'D_\epsilon\ \mathbf{1}'D_\epsilon\ \ldots \mathbf{1}'D_\epsilon]\hat{\alpha}) \\ &= \frac{\mathbf{1}}{n}(\mathbf{1}'x(0) + t\sum_{i=1}^{n}\epsilon_i). \end{aligned}$$

By construction $\hat{x}^{\mathbf{1}}(t) \in \mathcal{X}_C^{\mathbf{1}}(t)$, and $\lim_{t\to\infty} \left\|\hat{x}^{\mathbf{1}}(t)\right\|_2 = +\infty$, which concludes the proof. □

Proposition 1 states that in case of constant weights, when consensus is achieved, the consensus value is not necessarily bounded. This means that even when the difference among the agents' state vanishes, the state of each agent can diverge.

In order to derive asymptotic bounds on the quantity $r_C(t)$, defined as in (14), let us consider the dynamics of the projection of $x(t)$ on the subspace orthogonal to $\mathbf{1}$. Let $x^{\mathbf{1}^\perp}(t) \triangleq P^{\mathbf{1}^\perp}x(t)$. From (16) and (17) it follows

$$\begin{aligned} x^{\mathbf{1}^\perp}(t+1) &= P^{\mathbf{1}^\perp}Fx^{\mathbf{1}^\perp}(t) + P^{\mathbf{1}^\perp}v(t) \\ x^{\mathbf{1}^\perp}(0) &= P^{\mathbf{1}^\perp}x_0 \end{aligned}. \tag{18}$$

Some preliminary results are needed.

Lemma 1. *F is a primitive matrix. Moreover, $\lambda_1(F) = 1$ is a simple eigenvalue of F, with eigenvector $\mathbf{1}$, and $|\lambda_i(F)| < 1$, $i = 2, \ldots, n$.*

Being $F = I + W$, the following result follows directly from Lemma 1.

Corollary 1. *$\lambda_1(W) = 0$ is a simple eigenvalue of W, with eigenvector $\mathbf{1}$, and $|\lambda_i(W)+1| < 1$, $i = 2, \ldots, n$.*

Lemma 2. *The norm of matrix $P^{\mathbf{1}^\perp} F$ is $\left\| P^{\mathbf{1}^\perp} F \right\|_2 = \sigma_2(F) < 1$.*

We are now ready to present an upper bound on the asymptotic disagreement of the team, in case of constant weights.

Theorem 1. *Let $x^{\mathbf{1}^\perp}(t)$ evolve according to (18). Then*

$$\lim_{t\to\infty} \left\| x^{\mathbf{1}^\perp}(t) \right\|_2 \leq \frac{\overline{\delta}}{1 - \sigma_2(F)},$$

where $\overline{\delta} = \max_{v \in D_\epsilon \mathcal{B}_\infty} \left\| P^{\mathbf{1}^\perp} v \right\|_2$.

Proof: From (16) and (18) it follows

$$\begin{aligned} \left\| x^{\mathbf{1}^\perp}(t+1) \right\|_2 &= \left\| P^{\mathbf{1}^\perp} F x^{\mathbf{1}^\perp}(t) + P^{\mathbf{1}^\perp} v(t) \right\|_2 \leq \left\| P^{\mathbf{1}^\perp} F \right\|_2 \left\| x^{\mathbf{1}^\perp}(t) \right\|_2 + \overline{\delta} \\ &= \sigma_2(F) \left\| x^{\mathbf{1}^\perp}(t) \right\|_2 + \overline{\delta} \end{aligned}$$

where the last equality comes from Lemma 2. Now, if we consider the system

$$\begin{cases} z(t+1) = \sigma_2(F) z(t) + \overline{\delta} \\ z(0) = \left\| x^{\mathbf{1}^\perp}(0) \right\|_2 \end{cases},$$

then $\left\| x^{\mathbf{1}^\perp}(t) \right\|_2 \leq z(t)$, $t \geq 0$. The result follows by noting that $\lim_{t\to\infty} z(t) = \dfrac{\overline{\delta}}{1 - \sigma_2(F)}$ since $0 \leq \sigma_2(F) < 1$ by Lemma 2. □

The following lemma is needed in order to obtain a lower bound on the asymptotic disagreement of the team.

Lemma 3. *Let $x^{\mathbf{1}^\perp}(t)$ evolve according to (18) and let $v(t) = v$, $t \geq 0$. Then*

$$\lim_{t\to\infty} x^{\mathbf{1}^\perp}(t) = (I - P^{\mathbf{1}^\perp} F)^{-1} P^{\mathbf{1}^\perp} v.$$

Theorem 2. *Let $x^{\mathbf{1}^\perp}(t)$ evolve according to (18). Then there exists a feasible noise realization $v(t)$ such that*

$$\lim_{t\to\infty} \left\| x^{\mathbf{1}^\perp}(t) \right\|_2 \geq \max_{i=1,\ldots,n-1} \frac{\underline{\delta}_i}{\sigma_i(W)}$$

where $\underline{\delta}_i \triangleq \max_{v \in D_\epsilon \mathcal{B}_\infty} |Y_i^{W'} v|$ and $W = Y^W \Sigma^W U^{W'}$ is a SVD of W.

Proof:

Consider the following SVDs

$$\bar{F} \triangleq (I - P^{\mathbf{1}^\perp} F)^{-1} = \bar{Y}\bar{\Sigma}\bar{U}', \tag{19}$$

$$\tilde{F} \triangleq I - P^{\mathbf{1}^\perp} F = \tilde{Y}\tilde{\Sigma}\tilde{U}', \tag{20}$$

$$W = Y^W \Sigma^W U^{W\prime}. \tag{21}$$

Clearly the following relationships hold

$$\bar{Y} = \tilde{U}, \quad \bar{\Sigma} = \tilde{\Sigma}^{-1}, \quad \bar{U} = \tilde{Y}. \tag{22}$$

From Assumption 3, the smallest singular value of matrix W is $\sigma_n(W) = 0$ and the corresponding left and right singular vectors are $Y_n^W = U_n^W = \dfrac{\mathbf{1}}{\sqrt{n}}$. Now, note that from (16)

$$\tilde{F} = I - P^{\mathbf{1}^\perp} F = \frac{\mathbf{1}\mathbf{1}'}{n} - W = \frac{\mathbf{1}\mathbf{1}'}{n} - \sum_{i=1}^{n-1} \sigma_i(W) Y_i^W U_i^{W\prime}. \tag{23}$$

Hence, from (20), (21) and (23) it follows for $i = 1, \ldots, n-1$,

$$\tilde{Y} = \left[Y_1^W \ldots Y_{n-1}^W \ \frac{\mathbf{1}}{\sqrt{n}} \right], \quad \tilde{U} = \left[-U_1^W \cdots - U_{n-1}^W \ \frac{\mathbf{1}}{\sqrt{n}} \right], \quad \sigma_i(\tilde{F}) = \sigma_i(W), \tag{24}$$

and $\sigma_n(\tilde{F}) = 1$. Exploiting (22), one finally gets for $i = 1, \ldots, n-1$,

$$\bar{Y} = \left[-U_1^W \cdots - U_{n-1}^W \ \frac{\mathbf{1}}{\sqrt{n}} \right], \quad \bar{U} = \left[Y_1^W \ldots Y_{n-1}^W \ \frac{\mathbf{1}}{\sqrt{n}} \right], \quad \sigma_i(\bar{F}) = \frac{1}{\sigma_i(W)}, \tag{25}$$

and $\sigma_n(\bar{F}) = 1$. Note that $\sigma_i(\tilde{F})$ and $\sigma_i(\bar{F})$ are now not necessarily arranged in decreasing order. Let

$$\underline{\delta}_i \triangleq \max_{v \in D_\epsilon \mathcal{B}_\infty} |Y_i^{W\prime} v| = Y_i^{W\prime} D_\epsilon \text{sgn}\ (Y_i^W), \ i = 1, \ldots, n-1, \tag{26}$$

and consider the corresponding noise realizations

$$v^{\{i\}}(t) = v^{\{i\}} \triangleq D_\epsilon \text{sgn}\ (Y_i^W) \quad t \geq 0, \quad i = 1, \ldots, n-1. \tag{27}$$

Note that $v^{\{i\}}$ is feasible by construction. Since Y_i^W, $i = 1, \ldots, n-1$ are orthogonal to $\mathbf{1}$ because Y^W is unitary, $P^{\mathbf{1}^\perp} v^{\{i\}}$ can be rewritten as

$$P^{\mathbf{1}^\perp} v^{\{i\}} = \sum_{j=1}^{n-1} \beta_{ij} Y_j^W \tag{28}$$

for some $\beta_{ij} \in \mathbb{R}$. Note that by (26)-(27), it holds $\beta_{ii} = \underline{\delta}_i$, for each i. Then, from (19), (25), and (28), one gets

$$\left\| (I - P^{\mathbf{1}^\perp} F)^{-1} P^{\mathbf{1}^\perp} v^{\{i\}} \right\|_2^2 = \left\| \bar{Y} \bar{\Sigma} \bar{U}' \left(\sum_{j=1}^{n-1} \beta_{ij} Y_j^W \right) \right\|_2^2 = \left\| \sum_{j=1}^{n-1} \sigma_j(\bar{F}) \beta_{ij} U_j^W \right\|_2^2$$
$$= \sum_{j=1}^{n-1} \left(\frac{\beta_{ij}}{\sigma_j(W)} \right)^2 \geq \left(\frac{\underline{\delta}_i}{\sigma_i(W)} \right)^2.$$

The thesis follows from Lemma 3. □

From Theorems 1 and 2, it can be concluded that the set $\mathcal{X}_C^{\mathbf{1}^\perp}(t)$ is bounded for all t. Moreover, $r_C(t)$ in (14) satisfies

$$\max_{i=1,\dots,n-1} \frac{\underline{\delta}_i}{\sigma_i(W)} \leq \lim_{t\to\infty} r_C(t) \leq \frac{\overline{\delta}}{1 - \sigma_2(F)}, \tag{29}$$

where $\underline{\delta}_i = Y_i^{W\prime} D_\epsilon \text{sgn}\,(Y_i^W)$ and $\overline{\delta} = \max_{v \in D_\epsilon \mathcal{B}_\infty} \left\| P^{\mathbf{1}^\perp} v \right\|_2$.

4.2. Vanishing weights

Let us now turn our attention to consensus protocols featuring vanishing weights. To this purpose, consider the LTV system

$$\begin{array}{r}x(t+1) = F(t)x(t) + a(t)v(t)\\ x(0) = x_0\end{array}, \tag{30}$$

where $F(t) = I + a(t)W$, and the weight matrix W is chosen as in (15). From now on, the following assumption is made on the weighting sequence $a(t)$.

Assumption 4 (Forgetting factor). *The sequence $a(t) \in \mathbb{R}$ satisfies*

$$0 < a(t) < \frac{1}{\max_i |w_{ii}|}, \quad t \geq 0, \tag{31}$$

$$\lim_{t\to\infty} a(t) = 0, \tag{32}$$

$$\sum_{t=0}^{\infty} a(t) = +\infty. \tag{33}$$

Examples of such functions are $a(t) = 1/(t+\gamma)^r$, $\gamma > 1$, $0 < r \leq 1$. Recall that if $\sum_{t=0}^{\infty} a(t) < +\infty$, then asymptotic consensus is not guaranteed any more in the noiseless case [12].

Proposition 2. *Let $a(t)$ satisfy* (31)-(33). *Then, the feasible state set $\mathcal{X}_V$ given by* (12) *is asymptotically unbounded along the direction identified by the vector* $\mathbf{1}$.

Proof: Since, under Assumption 3, $\mathbf{1}'F(t) = \mathbf{1}'$, from (13) it follows

$$\mathbf{1}'\Phi(t_2, t_1) = \mathbf{1}', \quad 0 \le t_1 < t_2. \tag{34}$$

Denote by $\mathcal{X}_V^{\mathbf{1}}(t)$ the orthogonal projection of $\mathcal{X}_V(t)$ in (12) on the subspace spanned by $\mathbf{1}$, and consider the state $\hat{x}(t) \in \mathcal{X}_V(t)$ corresponding to $\hat{\alpha} = [1\ 1\ ...\ 1]' \in \mathbb{R}^{tn}$ in (12). Then, from (12)-(13) and exploiting (34), its projection $\hat{x}^{\mathbf{1}}(t)$ on $\mathbf{1}$ is given by

$$\hat{x}^{\mathbf{1}}(t) = \frac{\mathbf{1}}{n}(\mathbf{1}'x(0) + \sum_{j=1}^{n} \epsilon_j \sum_{i=0}^{t-1} a(i)).$$

By construction $\hat{x}^{\mathbf{1}}(t) \in \mathcal{X}_V^{\mathbf{1}}(t)$ and its norm tends to infinity by Assumption 4, which concludes the proof. □

Proposition 2 shows that, also in case of vanishing weights, the consensus value is not necessarily bounded.

In order to get asymptotic bounds on the quantity $r_V(t)$ in (14), let us consider the dynamics of the projection of $x(t)$ on the subspace orthogonal to $\mathbf{1}$. From (30) it follows

$$\begin{aligned} x^{\mathbf{1}^\perp}(t+1) &= P^{\mathbf{1}^\perp} F(t) x^{\mathbf{1}^\perp}(t) + a(t) P^{\mathbf{1}^\perp} v(t) \\ x^{\mathbf{1}^\perp}(0) &= P^{\mathbf{1}^\perp} x_0 \end{aligned}. \tag{35}$$

The following lemmas are instrumental in establishing asymptotic upper and lower bounds to r_V.

Lemma 4. *Let $z(t) \in \mathbb{R}$ evolve according to*

$$z(t+1) = (1 - ca(t))z(t) + a(t)\delta, \tag{36}$$

where $a(t)$ satisfies Assumption 4, $c > 0$ and $\delta \ge 0$. Then

$$\lim_{t\to\infty} z(t) = \frac{\delta}{c}.$$

Lemma 5. *Let $c_1 = -\lambda_2(W+W') > 0$, $c_2 = \lambda_1(W'W) > 0$, $c_3 = -\lambda_2(W+W'+W'W) > 0$, $F(t) = I + a(t)W$, and $a(t)$ satisfy Assumption 4. Then, there exists a finite time $\hat{t}$ such that*

$$0 < a(t)(c_1 - c_2 a(t)) < 1, \qquad \forall t \ge \hat{t}, \tag{37}$$

and

$$\left\| P^{\mathbf{1}^\perp} F(t) x \right\|_2^2 \le [1 - a(t)(c_1 - c_2 a(t))] \|x\|_2^2 \le [1 - c_3 a(t)] \|x\|_2^2, \quad \forall t \ge \hat{t}, \tag{38}$$

for any $x \in \mathbb{R}^n$ such that $\mathbf{1}'x = 0$.

The following theorem gives an upper bound to the asymptotic disagreement of the team, in case of vanishing weights.

Theorem 3. *Let $x^{\mathbf{1}^\perp}(t)$ evolve according to* (35)*. Then*

$$\lim_{t\to\infty} \left\| x^{\mathbf{1}^\perp}(t) \right\|_2 \leq -\frac{2\overline{\delta}}{\lambda_2(W+W')}$$

where $\overline{\delta} = \max_{v\in D_\epsilon \mathcal{B}_\infty} \left\| P^{\mathbf{1}^\perp} v \right\|_2$.

Proof: From the definition of $\overline{\delta}$ and from equation (35) it follows

$$\left\| x^{\mathbf{1}^\perp}(t+1) \right\|_2 \leq \left\| P^{\mathbf{1}^\perp} F(t) x^{\mathbf{1}^\perp}(t) \right\|_2 + a(t)\overline{\delta}.$$

Since $\mathbf{1}' x^{\mathbf{1}^\perp}(t) = 0$, from Lemma 5 one gets

$$\left\| x^{\mathbf{1}^\perp}(t+1) \right\|_2 \leq [1 - a(t)(c_1 - c_2 a(t))]^{1/2} \left\| x^{\mathbf{1}^\perp}(t) \right\|_2 + a(t)\overline{\delta}, \quad t \geq \hat{t}, \tag{39}$$

where $c_1 = -\lambda_2(W+W')$ and $c_2 = \lambda_1(W'W)$. Recall that for $0 < x < 1$, $(1-x)^{1/2} \leq 1 - \frac{1}{2}x$. Then, from (37), inequality (39) becomes

$$\left\| x^{\mathbf{1}^\perp}(t+1) \right\|_2 \leq \left(1 - a(t)\frac{c_1 - c_2 a(t)}{2}\right) \left\| x^{\mathbf{1}^\perp}(t) \right\|_2 + a(t)\overline{\delta}, \quad t \geq \hat{t}. \tag{40}$$

For all $\epsilon > 0$ there exists a time instant $t(\epsilon)$ such that $c_2 a(t) \leq \epsilon$, $t \geq t(\epsilon)$ and consequently

$$\left(1 - a(t)\frac{c_1 - c_2 a(t)}{2}\right) \leq \left(1 - a(t)\frac{c_1 - \epsilon}{2}\right), \quad t \geq t(\epsilon).$$

Hence from (40)

$$\left\| x^{\mathbf{1}^\perp}(t+1) \right\|_2 \leq \left(1 - a(t)\frac{c_1 - \epsilon}{2}\right) \left\| x^{\mathbf{1}^\perp}(t) \right\|_2 + a(t)\overline{\delta}, \quad t \geq t(\epsilon),$$

which, combined with Lemma 4, implies

$$\lim_{t\to\infty} \left\| x^{\mathbf{1}^\perp}(t) \right\|_2 \leq \frac{2\overline{\delta}}{c_1 - \epsilon} = -\frac{2\overline{\delta}}{\lambda_2(W+W') + \epsilon}$$

The thesis follows from the arbitrariness of ϵ. □

The following lemma is instrumental in establishing a lower bound on the asymptotic disagreement of the team.

Lemma 6. *Let $x^{\mathbf{1}^\perp}(t)$ evolve according to* (35)*, and let $v(t) = v$, $t \geq 0$. Then*

$$\lim_{t\to\infty} x^{\mathbf{1}^\perp}(t) = (I - P^{\mathbf{1}^\perp} F)^{-1} P^{\mathbf{1}^\perp} v.$$

By noting that Lemma 6 gives the same limit as Lemma 3, the next result stems directly from Theorem 2.

Theorem 4. *Let $x^{\mathbf{1}^\perp}(t)$ evolve according to* (35)*. Then there exists a feasible noise realization $v(t)$ such that*

$$\lim_{t\to\infty} \left\| x^{\mathbf{1}^\perp}(t) \right\|_2 \geq \max_{i=1,\ldots,n-1} \frac{\underline{\delta}_i}{\sigma_i(W)}$$

where $\underline{\delta}_i \triangleq \max_{v \in D_\epsilon \mathcal{B}_\infty} |Y_i^{W'} v|$, and $W = Y^W \Sigma^W U^{W'}$ is a SVD of W.

Summarizing, from the results in Theorems 3 and 4, one can conclude that the set $\mathcal{X}_V^{\mathbf{1}^\perp}(t)$ is bounded for all t. Moreover, $r_V(t)$ in (14) satisfies

$$\max_{i=1,\ldots,n-1} \frac{\underline{\delta}_i}{\sigma_i(W)} \leq \lim_{t\to\infty} r_V(t) \leq -\frac{2\overline{\delta}}{\lambda_2(W+W')}, \tag{41}$$

where $\underline{\delta}_i = Y_i^{W'} D_\epsilon \mathrm{sgn}\,(Y_i^W)$ and $\overline{\delta} = \max_{v \in D_\epsilon \mathcal{B}_\infty} \left\| P^{\mathbf{1}^\perp} v \right\|_2$.

By comparing the results of Theorems 2 and 4, one finds that the lower bounds on the asymptotic value of $r_C(t)$ and $r_V(t)$ are the same. On the contrary, Theorems 1 and 3 provides upper bounds on the asymptotic disagreement $r_C(t)$ and $r_V(t)$ which are different, in general. It is natural to wonder whether there is a relationship between the two upper bounds. The next lemma provides the answer to this question.

Lemma 7. *Let $F = I + W$, where W is defined as in* (15)*. Then*

$$\frac{1}{1-\sigma_2(F)} \geq -\frac{2}{\lambda_2(W+W')}.$$

Lemma 7 implies that the upper bound on the asymptotic value of $r_V(t)$ is smaller than or equal to the upper bound found for the asymptotic value of $r_C(t)$.

It is worth remarking that the upper and lower bounds obtained in Theorems 1-4 are valid for all possible *time-varying* sequences $v(t)$ satisfying the condition $v(t) \in D_\epsilon \mathcal{B}_\infty$, stemming from the UBB noise assumption (7). A constant noise realization is used only to derive instrumental theoretical results, like Lemma 3 and Lemma 6.

4.3. Symmetric weight matrix

The results (29) and (41) take on a special form when the weight matrix W in (15) is symmetric. Let as usual $\sigma_i(F)$ and $\lambda_i(F)$ denote the singular values and the eigenvalues, respectively, of matrix F, arranged in decreasing order. Being F symmetric, $\sigma_i(F) = |\lambda_{j_i}(F)|$, for $i = 1, \ldots, n$ and for some $j_i \in \{1, \ldots, n\}$. Then, according to Lemma 1, $\sigma_1(F) = \lambda_1(F) = 1$ and

$$\sigma_2(F) = \max_{i=2,\ldots,n} |\lambda_i(F)| = \max\{\lambda_2(F), -\lambda_n(F)\}.$$

Moreover, $\lambda_2(W+W') = 2\lambda_2(W)$. Let e_i be a unitary eigenvector associated to $\lambda_i(F)$. From the definition of F, clearly $We_i = (\lambda_i(F)-1)e_i$. When W is symmetric, the input and output singular vectors Y_i^W, U_i^W correspond (up to a sign and a reordering) to its eigenvectors

$$Y_i^W = e_{n-i+1}, \quad U_i^W = -e_{n-i+1}, \quad i = 1, \ldots, n.$$

Hence $\sigma_i(W) = -\lambda_{n-i+1}(W) = 1 - \lambda_{n-i+1}(F)$ and

$$\underline{\delta}_i = Y_i^{W'} D_\epsilon \text{sgn}\,(Y_i^W) = e_{n-i+1} D_\epsilon \text{sgn}\,(e_{n-i+1}).$$

The above discussion leads to the following corollaries.

Corollary 2. *Let W be a symmetric matrix. Then, the set $\mathcal{X}_C^{\mathbf{1}^\perp}(t)$ is bounded for all t. Moreover, $r_C(t)$ in (14) satisfies*

$$\max_{i=2,\dots,n} \frac{\tilde{\underline{\delta}}_i}{1-\lambda_i(F)} \le \lim_{t\to\infty} r_C(t) \le \frac{\overline{\delta}}{1-\lambda_M(F)}, \tag{42}$$

where $\tilde{\underline{\delta}}_i = e_i D_\epsilon \text{sgn}\,(e_i)$, $\overline{\delta} = \max_{v\in D_\epsilon \mathcal{B}_\infty} \left\| P^{\mathbf{1}^\perp} v \right\|_2$ and $\lambda_M(F) = \max\{\lambda_2(F), -\lambda_n(F)\}$.

Corollary 3. *Let W be a symmetric matrix, and let $a(t)$ satisfy Assumption 4. Then, the set $\mathcal{X}_V^{\mathbf{1}^\perp}(t)$ is bounded for all t. Moreover, $r_V(t)$ in (14) satisfies*

$$\max_{i=2,\dots,n} \frac{\tilde{\underline{\delta}}_i}{1-\lambda_i(F)} \le \lim_{t\to\infty} r_V(t) \le \frac{\overline{\delta}}{1-\lambda_2(F)}, \tag{43}$$

where $\tilde{\underline{\delta}}_i = e_i D_\epsilon \text{sgn}\,(e_i)$ and $\overline{\delta} = \max_{v\in D_\epsilon \mathcal{B}_\infty} \left\| P^{\mathbf{1}^\perp} v \right\|_2$.

The bounds provided by the previous corollaries are exactly the same found in [17], except for the lower bound in (42). In fact, in [17], a non-constant noise realization $v(t)$ has been found, giving rise to the (generally tighter) lower bound

$$\max_{i=2,\dots,n} \frac{\tilde{\underline{\delta}}_i}{1-|\lambda_i(F)|} \le \lim_{t\to\infty} r_C(t). \tag{44}$$

Clearly, if the maximum in (44) is attained in correspondence of a positive $\lambda_i(F)$, then the two bounds coincide.

4.4. Discussion

For the classes of consensus algorithms considered, it turns out that if the updating scheme ensures the achievement of consensus in the noise-free scenario, then the feasible state set is asymptotically contained in an infinite cylinder aligned with $\mathbf{1}$, and whose radius can be bounded by functions of the singular values (or eigenvalues) of the weight matrix and of the maximum amplitude of the measurement errors. It is worth remarking that the radius does not depend on the initial state of the agents. This means that the maximum deviation from consensus, with respect to all possible noise realizations, is independent of the initial disagreement of the team.

Identical lower bounds have been found for constant and vanishing weights. Concerning the upper bound, the one found in case of vanishing weights is smaller than or equal to its counterpart in case of constant weights (Lemma 7). This means that an algorithm like (6) provides some improvement also in a worst-case scenario, at least in principle. However, whenever matrix W is symmetric and such that $\lambda_M(F) = \lambda_2(F)$, the upper bound is the same for both classes (Corollaries 2 and 3). Moreover, it depends only on the second largest eigenvalue of matrix F. It is known that such an eigenvalue determines the convergence rate to the consensus value in absence of measurement noise (the smaller meaning the faster, see [6, 18]). Hence, a faster mixing network guarantees also a smaller worst-case asymptotic difference among the agents' states.

It is interesting to observe that the upper bound in case of vanishing weights does not depend on the rate of convergence of the weighting sequence $a(t)$, as long as it satisfies (33). Note that if $a(t)$ is selected such that $\sum_{t=0}^{\infty} a(t) < +\infty$, then even if $v(t) = 0$, $\forall t \geq 0$, i.e. in the noise-free scenario, consensus cannot be reached. Nonetheless, an $a(t)$ such that its summation converge, would have the advantage of bounding the feasible state set also along the direction $\mathbf{1}$ (see the proof of Proposition 2). Hence, in a worst-case analysis it could be of interest to choose a faster vanishing $a(t)$, in order to trade off boundedness of the agents' states and maximum asymptotic disagreement of the team.

Finally, it is worth remarking that the main reason why protocols like (6) do not guarantee consensus in a set-membership framework (differently from what happens when measurement noise is modeled in a stochastic setting, see [12]) is that the noise is only assumed to be bounded, and biased noise realizations are allowed as well.

5. Numerical results

This section presents a numerical example, illustrating the main theoretical results described so far. The team is composed of 10 agents, connected through a communication graph generated similarly to what has been done in [18]. The network topology has been obtained by randomly distributing the positions of the agents in a 1×1 box in the plane, and connecting two agents if and only if their distance is smaller than 0.5. The resulting network, composed of 31 edges, is connected, thus satisfying Assumption 2. The weight matrix W has been randomly generated according to equation (15) and such that Assumption 3 is satisfied. The measurement noise η_{ij} is assumed to be bounded as in (7), with $\bar{\epsilon} = 0.1$. The forgetting factor $a(t)$ adopted in the updating rule (6) is chosen as $a(t) = \frac{1}{(t+5)^{0.6}}$, thus satisfying Assumption 4. Within this setting, the asymptotic bounds (29) on the disagreement in case of constant weights become $\underline{r}_C \leq \lim_{t\to\infty} r_C(t) \leq \overline{r}_C$, where $\underline{r}_C = 0.49$ and $\overline{r}_C = 0.85$. In case of vanishing weights, the bounds (41) take on the value $\underline{r}_V \leq \lim_{t\to\infty} r_V(t) \leq \overline{r}_V$, where $\underline{r}_V = 0.49$ and $\overline{r}_V = 0.62$. In this example, the updating scheme based on vanishing weights provides a smaller upper bound than the constant weights scheme, $\overline{r}_V < \overline{r}_C$, as pointed out in Section 4.4.

Figures 1-(a) and 1-(b) summarize the results of 1000 simulation runs, starting from initial conditions $x_i(0)$ randomly generated in the interval $[-10,\ 10]$. The average and the maximum of $\left\| x^{\mathbf{1}^{\perp}}(t) \right\|_2$ are depicted at each time instant, for both choices of the updating rule. The asymptotic upper bounds $\overline{r}_{(\cdot)}$ (dash-dotted line) are also shown. The measurement noise $\eta_{ij}(t)$ is uniformly distributed in the interval $[-\bar{\epsilon},\ \bar{\epsilon}]$. When constant weights are used,

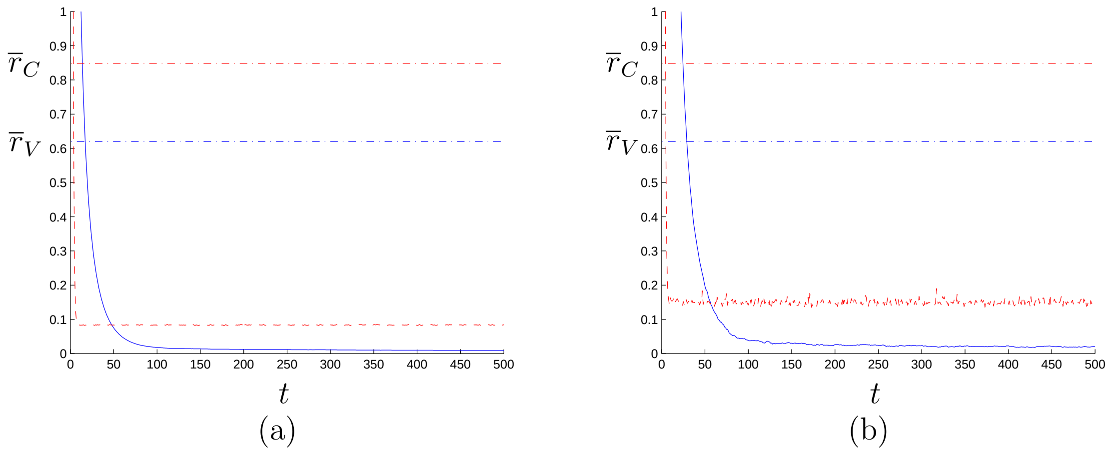

Figure 1: Average value (a) and maximum (b) of $\left\|x^{\mathbf{1}^\perp}(t)\right\|_2$, over 1000 simulation runs, in case of constant weights (dashed line) and vanishing weights (solid line).

the average of $\left\|x^{\mathbf{1}^\perp}(t)\right\|_2$ settles around 0.085, while the maximum values oscillate around 0.15 (Figure 1). In case of vanishing weights, the same quantities are one order of magnitude smaller. It can be noticed that in general constant weights protocols feature faster transient at the expense of larger steady-state error. In the previous example, the theoretical upper bound on the asymptotic value of $r(t)$ turns out to be quite conservative. However, it is worth recalling that such a bound holds for all possible noise realizations satisfying (7). Even though the worst-case noise $\eta_{ij}(t)$ is not easy to determine, some unfavorable realizations can be figured out. In Figure 2, $\left\|x^{\mathbf{1}^\perp}(t)\right\|_2$ is shown when $\eta_{ij}(t)$ is generated such that $v(t) = v^{\{i^*\}}$ as defined in equation (27), and i^* is the index of the largest ratio $\frac{\delta_i}{\sigma_i(W)}$ (see Theorems 2 and 4), i.e.

$$i^* = \arg \max_{i=1,\dots,n-1} \frac{\delta_i}{\sigma_i(W)}.$$

For this noise realization, the actual value of $\left\|x^{\mathbf{1}^\perp}(t)\right\|_2$ is eventually larger than $\underline{r}_{(\cdot)}$ (lower dash-dotted line in Figure 2), and the final value is the same for both choices of the weights.

6. Conclusions

In this paper, the asymptotic properties of two classes of linear consensus algorithms have been analyzed, in presence of bounded measurement errors. The consensus protocols taken into account differ for the way the weighting matrix is chosen, being either constant over time or vanishing as time increases. Under the assumption of bounded errors the consensus problem has been formulated in a set-theoretic framework. By studying the evolution of the feasible state set, a worst-case analysis on the asymptotic disagreement of the team has been performed.

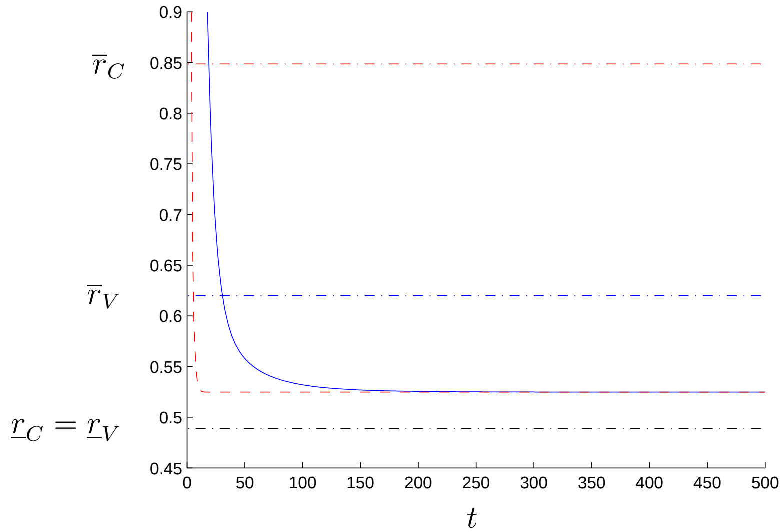

Figure 2: Value of $\left\|x^{\mathbf{1}^\perp}(t)\right\|_2$ for noise realization (27), in case of constant weights (dashed line) and vanishing weights (solid line).

It has been shown that for both kinds of algorithms, consensus cannot be guaranteed with respect to all possible noise realizations, but the difference among the agents' states is asymptotically bounded. Both upper and lower bounds have been derived, as a function of the bounds on the measurement noise and of the weight matrix.

There is a number of issues related to set-membership consensus which are going to be addressed in future work. One is the characterization of the noise realization giving rise to the maximum disagreement of the team, in order to achieve possible tighter bounds. Closely related to this topic is the synthesis of the weight matrix minimizing the worst-case asymptotic deviation from consensus. The extension of the results presented in this paper to communication networks with time-varying topology is also under investigation.

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A. Appendix

A.1. Proof of Lemma 1

By construction, F is non-negative and, from Assumption 2, the matrix F is irreducible (Theorem 6.2.24 in [19]). Since $F_{ii} > 0$, F is also aperiodic. Hence F is a primitive matrix and the Perron-Frobenius (PF) theorem applies (Theorem 1.1 in [20]). Hence, there exists a simple eigenvalue of maximum modulus $\lambda_1(F)$ with positive eigenvector. Moreover, $\lambda_1(F)$ lies between the minimum and maximum row sum of F (Corollary 1 in [20]). Since, by Assumption 3, $F\mathbf{1} = \mathbf{1}$, then $\lambda_1(F) = 1$ and $\mathbf{1}$ is the corresponding eigenvector.

A.2. Proof of Lemma 2

Consider a SVD of $F = Y^F \Sigma^F U^{F'}$. Since FF' has the same sparsity pattern of F^2, and F is primitive by Lemma 1, then F^2, and hence FF', is primitive as well, and the PF theorem applies. Hence $\lambda_1(FF') = 1$. Since, by Assumption 3, $F'\mathbf{1} = \mathbf{1}$, then $Y_1^F = U_1^F = \mathbf{1}/\sqrt{n}$, and $F = \frac{\mathbf{1}\mathbf{1}'}{n} + \sum_{i=2}^n \sigma_i(F) Y_i^F U_i^{F'}$. Hence, from (16), $P^{\mathbf{1}^\perp} F = \sum_{i=2}^n \sigma_i(F) Y_i^F U_i^{F'}$. In other words, $P^{\mathbf{1}^\perp} F$ has the same singular values of F, except for that at 1 (which is replaced by a singular value at 0). Thus $\left\| P^{\mathbf{1}^\perp} F \right\|_2 = \sigma_2(F) < 1$ where the inequality follows from the PF theorem applied to FF'.

A.3. Proof of Lemma 3

Recall that for any square matrix A, the spectral radius $\rho(A) \leq ||A||$, for any matrix norm $||\cdot||$ (Theorem 5.6.9 in [19]). Then from Lemma 2 one has $\rho(P^{\mathbf{1}^\perp}F) \leq \left\|P^{\mathbf{1}^\perp}F\right\|_2 < 1$, and the result follows from the asymptotic stability of system (18).

A.4. Proof of Lemma 4

Notice that Assumption 4 ensures that there exists a finite time t_0 such that $ca(t) < 1$, $t \geq t_0$. Moreover, the state $z_e = \frac{\delta}{c}$ is the only equilibrium point for the time-varying system (36). Consider the change of coordinates $z'(t) \triangleq z(t) - \frac{\delta}{c}$. Then, from (36) it follows $z'(t+1) = (1 - ca(t))z'(t)$, and hence $z'(t) = \left[\prod_{k=t_0}^{t-1}(1 - ca(k))\right] z'(t_0)$, $t \geq t_0$. It is known that an infinite product of the form $\prod_{k=0}^{\infty}(1 - \alpha(k))$, with $0 \leq \alpha(k) < 1$, converges to a non zero value if and only if $\sum_{k=0}^{\infty} \alpha(k) < +\infty$ (e.g., see result 0.252 in [21], p. 14). This implies that if $\sum_{k=0}^{\infty} \alpha(k) = +\infty$, then $\prod_{k=0}^{\infty}(1 - \alpha(k)) = 0$. Hence, from (33), $\lim_{t\to\infty} z'(t) = 0$, for all $z'(t_0)$. Since $z'(t) = 0$ implies $z(t) = \frac{\delta}{c}$, the thesis follows.

A.5. Proof of Lemma 5

Since F is primitive, then $F'F$ is primitive as well, and its eigenvalues satisfy

$$0 < \lambda_n(F'F) \leq \cdots \leq \lambda_2(F'F) < \lambda_1(F'F) = 1.$$

Define $S = W + W' + W'W$. Then, $S = F'F - I$ and hence

$$-1 < \lambda_n(S) \leq \cdots \leq \lambda_2(S) < \lambda_1(S) = 0.$$

Hence S is negative semidefinite, and $c_3 = -\lambda_2(S) > 0$. Since $W'W$ is positive semidefinite, and rank$(W'W) = n-1$ by Corollary 1, then $c_2 = \lambda_1(W'W) > 0$. It is known (see Corollary 4.3.3 in [19]) that given two symmetric matrices A, B, with B positive semidefinite, then $\lambda_i(A) \leq \lambda_i(A+B)$, $i = 1, \ldots, n$. Let $A = W + W'$ and $B = W'W$, so that $A + B = S$. Then $\lambda_i(W + W') \leq \lambda_i(W + W' + W'W) = \lambda_i(S)$, $i = 1, \ldots, n$, from which it follows $\lambda_2(W + W') \leq \lambda_2(S) < 0$. Hence $c_1 = -\lambda_2(W + W') > 0$ and $c_3 \leq c_1$. Given the special structure of matrices A, B and S, it can be shown that the last inequality holds strictly. In fact, by contradiction, suppose $c_3 = c_1$, i.e. $\lambda_2(A) = \lambda_2(S) \triangleq l$. Let v_i^A, v_i^S, $i = 1, \ldots, n$, be unitary eigenvectors associated to $\lambda_i(A)$ and $\lambda_i(S)$, respectively. Then

$$v_2^{A'} A v_2^A = v_2^{S'} S v_2^S = l. \tag{45}$$

Now, notice that, by Corollary 1, $\ker(A) = \ker(B) = \ker(S) = \text{Span}\{\mathbf{1}\}$. Hence, v_i^A, v_i^S, $i = 2, \ldots, n$, are all orthogonal to the vector $\mathbf{1}$ and it is possible to write

$$v_2^A = \sum_{i=2}^{n} \alpha_i v_i^S \tag{46}$$

for some α_i such that $\sum_{i=2}^{n} \alpha_i^2 = 1$. By substituting (46) into (45), and recalling the expression of A, B and S, one gets

$$\begin{aligned} l &= \left(\sum_{i=2}^{n} \alpha_i v_i^S\right)' A \left(\sum_{i=2}^{n} \alpha_i v_i^S\right) = \left(\sum_{i=2}^{n} \alpha_i v_i^S\right)' (S - B) \left(\sum_{i=2}^{n} \alpha_i v_i^S\right) \\ &= \left(\sum_{i=2}^{n} \alpha_i v_i^S\right)' S \left(\sum_{i=2}^{n} \alpha_i v_i^S\right) - \gamma \end{aligned}$$

where $\gamma = v_2^{A'} B v_2^A > 0$, since B is positive semidefinite and v_2^A is orthogonal to $\ker(B)$. By noting that

$$\left(\sum_{i=2}^{n} \alpha_i v_i^S\right)' S \left(\sum_{i=2}^{n} \alpha_i v_i^S\right) = \sum_{i=2}^{n} \alpha_i^2 \lambda_i(S)$$

it follows

$$\lambda_2(S) = l = \sum_{i=2}^{n} \alpha_i^2 \lambda_i(S) - \gamma \leq \left(\sum_{i=2}^{n} \alpha_i^2\right) \lambda_2(S) - \gamma = \lambda_2(S) - \gamma < \lambda_2(S).$$

Thus, it must be $c_3 < c_1$. Hence, Assumption 4 ensures that there exist two finite time instants $\bar{t}$, $\hat{t}$, $\hat{t} \geq \bar{t}$ such that

$$-c_1 + c_2 a(t) \leq -c_3, \qquad t \geq \bar{t}, \tag{47}$$

and

$$0 < a(t)(c_1 - c_2 a(t)) < 1, \qquad t \geq \hat{t},$$

which concludes the first part of the proof.

Let us now consider a vector x such that $\mathbf{1}'x = 0$. By recalling that $\lambda_1(W + W') = 0$ and $\lambda_i(W + W') < 0$, $i = 2, \ldots, n$, Corollary 1 ensures that $x'(W + W')x \leq \lambda_2(W + W') \|x\|_2^2$, and, from (47),

$$\begin{aligned} x'(W + W' + a(t)W'W)x &= x'(W + W')x + a(t)x'(W'W)x \\ &\leq \lambda_2(W + W') \|x\|_2^2 + a(t)\lambda_1(W'W) \|x\|_2^2, \\ &= (-c_1 + c_2 a(t)) \|x\|_2^2 \leq -c_3 \|x\|_2^2, \qquad t \geq \bar{t}. \end{aligned} \tag{48}$$

Since, by Assumption 3, $P^{\mathbf{1}^\perp} F(t) = I - \frac{\mathbf{1}\mathbf{1}'}{n} + a(t)W$, then

$$\begin{aligned} \left\| P^{\mathbf{1}^\perp} F(t)x \right\|_2^2 &= x'F'(t)P^{\mathbf{1}^\perp} P^{\mathbf{1}^\perp} F(t)x = x' \left[(I + a(t)W)'(I + a(t)W)\right] x \\ &= \|x\|_2^2 + a(t)x' \left[W + W' + a(t)W'W\right] x. \end{aligned}$$

Finally, from (37)- (48) one gets

$$\left\| P^{\mathbf{1}^\perp} F(t)x \right\|_2^2 \leq [1 - a(t)(c_1 - c_2 a(t))] \|x\|_2^2 \leq [1 - c_3 a(t)] \|x\|_2^2, \qquad t \geq \hat{t}.$$

A.6. Proof of Lemma 6

Being $F(t) = I + a(t)W$, one has

$$P^{\mathbf{1}^\perp} F(t) = I - (1 - a(t))\frac{\mathbf{11}'}{n} - a(t)\left(\frac{\mathbf{11}'}{n} - W\right). \tag{49}$$

Define $\eta(t) = x^{\mathbf{1}^\perp}(t) - (I - P^{\mathbf{1}^\perp} F)^{-1} P^{\mathbf{1}^\perp} v$, where $F = I + W$. Then

$$\begin{aligned}\eta(t+1) &= x^{\mathbf{1}^\perp}(t+1) - (I - P^{\mathbf{1}^\perp} F)^{-1} P^{\mathbf{1}^\perp} v \\ &= P^{\mathbf{1}^\perp} F(t) x^{\mathbf{1}^\perp}(t) + a(t) P^{\mathbf{1}^\perp} v - (I - P^{\mathbf{1}^\perp} F)^{-1} P^{\mathbf{1}^\perp} v \\ &= P^{\mathbf{1}^\perp} F(t)\eta(t) + \left(a(t)I - (I - P^{\mathbf{1}^\perp} F(t))(I - P^{\mathbf{1}^\perp} F)^{-1}\right) P^{\mathbf{1}^\perp} v.\end{aligned} \tag{50}$$

Since, from (16), $(I - P^{\mathbf{1}^\perp} F)^{-1} = \left(\frac{\mathbf{11}'}{n} - W\right)^{-1}$, by exploiting (49) one gets

$$\begin{aligned}a(t)I - (I - P^{\mathbf{1}^\perp} F(t))(I - P^{\mathbf{1}^\perp} F)^{-1} &= a(t)I - \left[(1 - a(t))\frac{\mathbf{11}'}{n} + a(t)\left(\frac{\mathbf{11}'}{n} - W\right)\right] \\ &\qquad \times \left(\frac{\mathbf{11}'}{n} - W\right)^{-1} \\ &= -(1 - a(t))\frac{\mathbf{11}'}{n}\left(\frac{\mathbf{11}'}{n} - W\right)^{-1} = -(1 - a(t))\frac{\mathbf{11}'}{n},\end{aligned} \tag{51}$$

where the last equality comes from the identity (see Corollary 5.6.16 in [19] and recall Lemma 2)

$$\left(\frac{\mathbf{11}'}{n} - W\right)^{-1} = (I - P^{\mathbf{1}^\perp} F)^{-1} = \sum_{k=0}^{\infty} (P^{\mathbf{1}^\perp} F)^k = \sum_{k=0}^{\infty} (P^{\mathbf{1}^\perp} + W)^k$$

and by noting that $\frac{\mathbf{11}'}{n}(P^{\mathbf{1}^\perp} + W)^k = 0$, if $k > 0$. Finally, substituting (51) in (50) one gets

$$\eta(t+1) = P^{\mathbf{1}^\perp} F(t)\eta(t). \tag{52}$$

Now consider the function $V(\eta, t) = \|\eta(t)\|_2^2$. Since $\mathbf{1}'\eta(t) = 0$, from (38) in Lemma 5 one gets

$$V(\eta, t+1) = \left\|P^{\mathbf{1}^\perp} F(t)\eta(t)\right\|_2^2 \leq (1 - c_3 a(t))V(\eta, t), \qquad t \geq \hat{t},$$

where $c_3 = -\lambda_2(W + W' + W'W)$. From Lemma 4 it follows $\lim_{t\to\infty} V(\eta, t) = 0$, which in turn implies $\lim_{t\to\infty} \eta(t) = 0$.

A.7. Proof of Lemma 7

Let us consider the matrix $A = \frac{F+F'}{2} = I + \frac{W+W'}{2}$. Being F primitive by Lemma 1, then A is primitive and all its row sums equal to 1. Hence

$$-1 < \lambda_n(A) \leq \cdots \leq \lambda_2(A) < \lambda_1(A) = 1, \tag{53}$$

and the unitary eigenvector corresponding to the largest eigenvalue is $\mathbf{1}/\sqrt{n}$. As a consequence, a SVD of A has the form

$$A = \frac{\mathbf{11}'}{n} + \sum_{i=2}^{n} \sigma_i(A) Y_i^A U_i^{A'}, \tag{54}$$

where $0 < \sigma_i(A) < 1$, $i = 2, \ldots, n$. Now consider the matrix $B = P^{\mathbf{1}^\perp} + \frac{W+W'}{2}$, where $P^{\mathbf{1}^\perp} = I - \frac{\mathbf{11}'}{n}$. Clearly, $B = A - \frac{\mathbf{11}'}{n}$. Moreover, $B\mathbf{1} = 0$ and hence a SVD of B is given by

$$B = \sum_{i=2}^{n} \sigma_i(A) Y_i^A U_i^{A'}. \tag{55}$$

By recalling the definition of A and B, equations (54)-(55) imply that

$$\sigma_2\left(I + \frac{W+W'}{2}\right) = \sigma_1\left(P^{\mathbf{1}^\perp} + \frac{W+W'}{2}\right). \tag{56}$$

By Lemma 2 and from equation (16), one has

$$\sigma_2(F) = \sigma_1(P^{\mathbf{1}^\perp} F) = \sigma_1(P^{\mathbf{1}^\perp} + W). \tag{57}$$

Putting together (56) and (57), one gets

$$\begin{aligned}\sigma_2\left(I + \frac{W+W'}{2}\right) &= \sigma_1\left(\frac{P^{\mathbf{1}^\perp} + W}{2} + \frac{P^{\mathbf{1}^\perp} + W'}{2}\right) \\ &\leq \sigma_1\left(\frac{P^{\mathbf{1}^\perp} + W}{2}\right) + \sigma_1\left(\frac{P^{\mathbf{1}^\perp} + W'}{2}\right) = \sigma_2(F)\end{aligned} \tag{58}$$

Looking at the spectrum of $I + \frac{W+W'}{2}$ (equation (53)), one concludes that

$$\begin{aligned}\sigma_2\left(I + \frac{W+W'}{2}\right) &= \max\left\{\lambda_2\left(I + \frac{W+W'}{2}\right), -\lambda_n\left(I + \frac{W+W'}{2}\right)\right\} \\ &\geq \lambda_2\left(I + \frac{W+W'}{2}\right) = 1 + \frac{1}{2}\lambda_2(W+W').\end{aligned} \tag{59}$$

Finally, from (58)-(59), one gets $\sigma_2(F) \geq 1 + \frac{1}{2}\lambda_2(W+W')$, from which the result easily follows.