Lecture 15
Subgradients

- subgradients and quasigradients
- subgradient calculus
- ellipsoid method for nondifferentiable problems

Motivation

extend ellipsoid method to

- nondifferentiable convex functions
- quasiconvex functions

idea: given x_k, we need to 'rule out' a halfspace at x_k, *i.e.*, find $g \neq 0$ s.t.

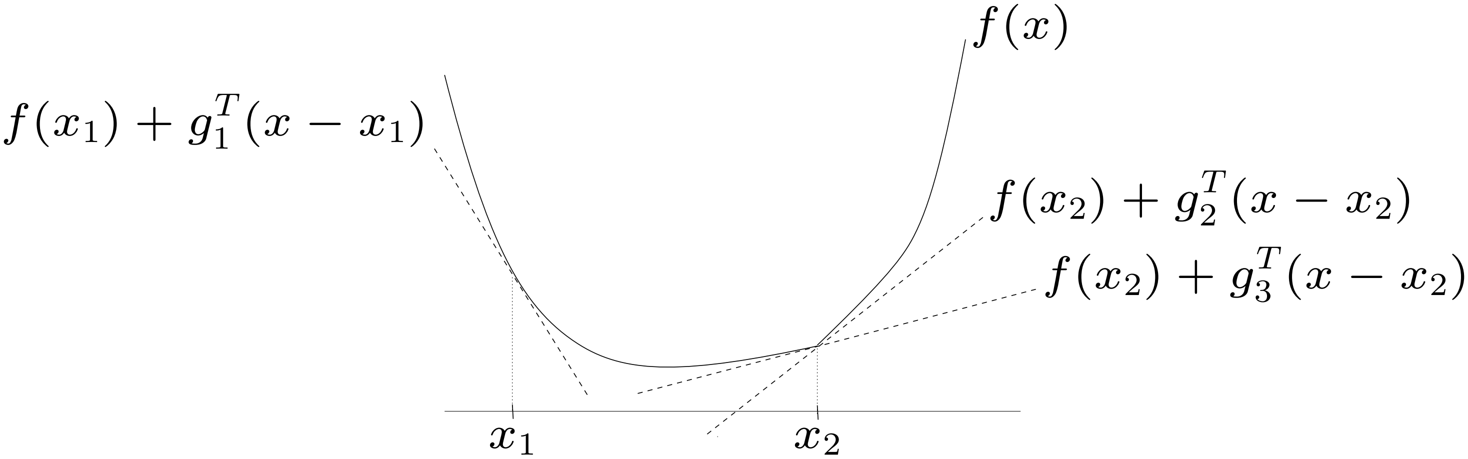
$$g^T(x^\star - x_k) \leq 0$$

- for differentiable fcts, g can be gradient
- but *any* such g will work . . .

Subgradient of a convex function

g is a *subgradient* of f at x if

$$f(y) \geq f(x) + g^T(y - x) \quad \text{for all } y$$



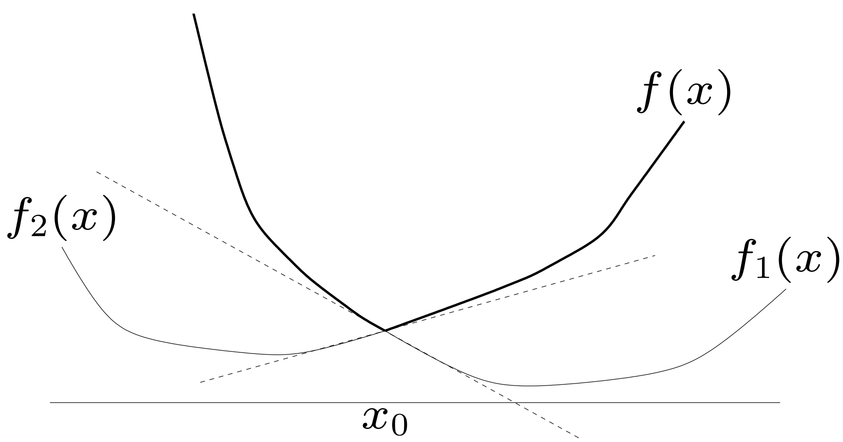
g_2, g_3 are subgradients at x_2; g_1 is a subgradient at x_1

- subgradient gives affine global lower bound on f
- $g^T(y - x) \geq 0 \Longrightarrow f(y) \geq f(x)$
- a convex function f is subdifferentiable ($i.e.$, at least one subgradient exists) at all points in $\mathbf{relint}\,\mathbf{dom}\, f$

Example

$$f = \max\{f_1, f_2\}$$

with f_1, f_2 convex and differentiable



- $f_1(x_0) > f_2(x_0)$: unique subgradient $g = \nabla f_1(x_0)$
- $f_2(x_0) > f_1(x_0)$: unique subgradient $g = \nabla f_2(x_0)$
- $f_1(x_0) = f_2(x_0)$: subgradients form a line segment

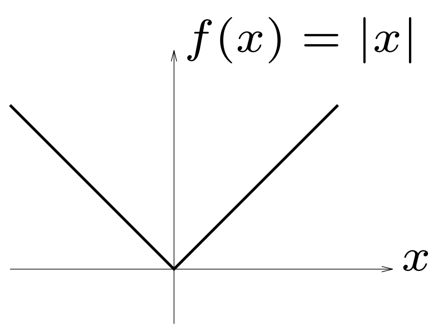
$$[\nabla f_1(x_0), \nabla f_2(x_0)]$$

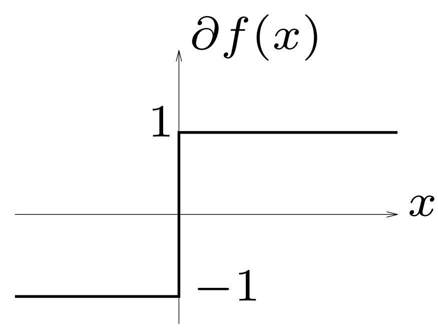
Subdifferentials

set of all subgradients of f at x is called the *subdifferential* of f at x, written $\partial f(x)$

- $\partial f(x)$ is a closed convex set
- $\partial f(x)$ nonempty (if f convex, and finite near x)
- $\partial f(x) = \{\nabla f(x)\}$ if f is differentiable at x
- if $\partial f(x) = \{g\}$, then f is differentiable at x and $g = \nabla f(x)$
- in most applications ($e.g.$, ellipsoid method), don't need complete $\partial f(x)$; it is sufficient to find one $g \in \partial f(x)$

example: $f(x) = |x|$





Calculus of subgradients

assumption: all functions are finite near x

- $\partial f(x) = \{\nabla f(x)\}$ if f is differentiable at x
- *scaling:* $\partial(\alpha f) = \alpha \partial f$ (if $\alpha > 0$)
- *addition:* $\partial(f_1 + f_2) = \partial f_1 + \partial f_2$
- *affine transformation of variables:*
  if $g(x) = f(Ax + b)$, then $\partial g(x) = A^T \partial f(Ax + b)$
- *pointwise maximum:* if $f = \max_{i=1,\ldots,m} f_i$, then

$$\partial f(x) = \mathbf{Co} \cup \{\partial f_i(x) \mid f_i(x) = f(x)\},$$

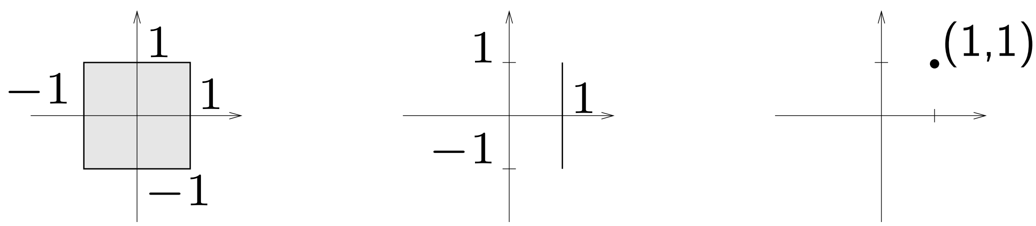
  $i.e.$, convex hull of union of subdifferentials of 'active' functions at x

  special case: if f_i differentiable

$$\partial f(x) = \mathbf{Co}\{\nabla f_i(x) \mid f_i(x) = f(x)\}$$

example

$$f(x) = \|x\|_1 = \max\{s^T x \mid s_i \in \{-1, 1\}\}$$



$\partial f(x)$ at $x = (0, 0)$ at $x = (1, 0)$ at $x = (1, 1)$

- *pointwise supremum:* if $f = \sup_{\alpha \in \mathcal{A}} f_\alpha$, then

$$\partial f_\beta(x) \subseteq \partial f(x)$$

if $f_\beta(x) = f(x)$ and $\beta \in \alpha$

(many technical conditions required for equality)

example

$$f(x) = \lambda_{\max}(A(x)) = \sup_{\|y\|=1} y^T A(x) y$$

where $A(x) = A(x)^T = A_0 + x_1 A_1 + \cdots + x_n A_n$

– $g_y(x) \overset{\Delta}{=} y^T A(x) y$ is affine in x, with

$$\nabla g_y(x) = (y^T A_1 y, \ldots, y^T A_n y)$$

– hence,

$$\partial f(x) = \mathbf{Co}\{\nabla g_y \mid A(x)y = \lambda_{\mathsf{max}}(A(x))y, \ \|y\| = 1\}$$

(not hard to verify)

- *minimization:* define $g(y)$ as the optimal value of

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \le y_i, \ \ i = 1, \ldots, m \end{array}$$

(f_i convex; variable x)

from duality (*cf.*, page 7-16):

$$g(y) \ge g(0) - \sum_{i=1}^{m} \lambda_i^\star y_i$$

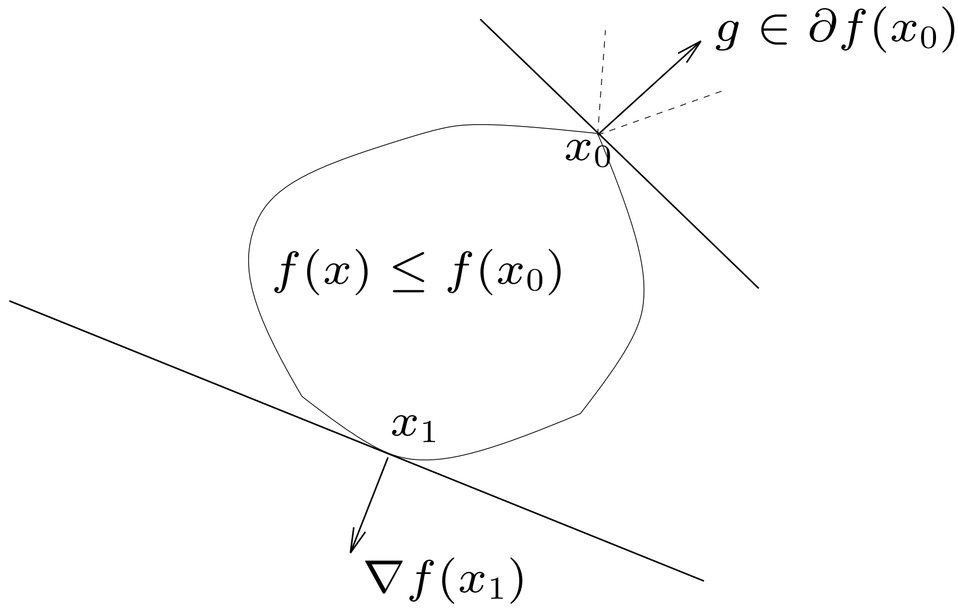
i.e., $-\lambda^\star$ is a subgradient of g at $y = 0$

Subgradients and sublevel sets

g is a subgradient at x if

$$f(y) \geq f(x) + g^T(y - x)$$

hence $f(y) \leq f(x) \Longrightarrow g^T(y - x) \leq 0$

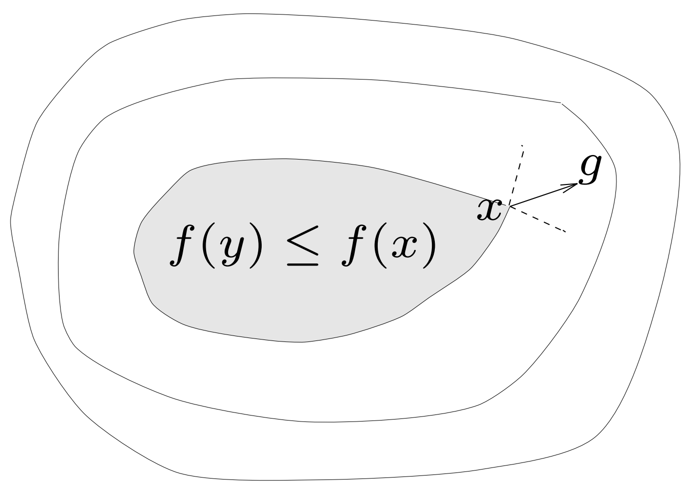


- f differentiable at x_0: $\nabla f(x_0)$ is normal to the sublevel set $\{x \mid f(x) \leq f(x_0)\}$
- f nondifferentiable at x_0: subgradient defines a supporting hyperplane to sublevel set through x_0

Quasigradients

if f is quasiconvex, then g is a *quasigradient* if

$$g^T(y-x) \geq 0 \Rightarrow f(y) \geq f(x)$$



- allows us to rule out a halfspace in the ellipsoid method
- quasigradients at x_0 form a cone

example

$$f(x) = \frac{a^Tx + b}{c^Tx + d}, \quad (\mathbf{dom}\, f = \{x | c^Tx + d > 0\})$$

$g = a - f(x_0)c$ is a quasigradient at x_0

proof: for $c^Tx + d > 0$:

$$a^T(x - x_0) \geq f(x_0)c^T(x - x_0) \Longrightarrow f(x) \geq f(x_0)$$

example: degree of $a_1 + a_2t + \cdots + a_nt^{n-1}$

$$f(a) = \min\{i \mid a_{i+2} = \cdots = a_n = 0\}$$

$g = \mathsf{sign}(a_{k+1})e_{k+1}$ (with $k = f(a)$) is a quasigradient at $a \neq 0$

proof:

$$g^T(b - a) = \mathsf{sign}(a_{k+1})b_{k+1} - |a_{k+1}| \geq 0$$

implies $b_{k+1} \neq 0$

Ellipsoid alg for nondifferentiable probs

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject} & f_i(x) \leq 0, \ i = 1, \dots, m \end{array}$$

f_i convex, $\mathbf{dom}\, f_i = \mathbf{R}^n$

given ellipsoid $\mathcal{E}$ with $x^\star \in \mathcal{E}$ if feasible, $\epsilon > 0$
repeat
  if for some j, $f_j(x) > 0$ ($i.e.$, x infeasible)
    find $g \in \partial f_j(x)$;
    if $f_j(x) + \inf_{z \in \mathcal{E}} g^T(z - x) > 0$, return(infeas);
  else ($i.e.$, x feasible)
    $u := f_0(x)$;
    find $g \in \partial f_0(x)$;
    $l := f_0(x) + \inf_{z \in \mathcal{E}} g^T(z - x)$;
    if $u - l < \epsilon$ return(x);
  $\mathcal{E}$:= min. volume ellipsoid containing
  $\mathcal{E} \cap \{z \mid g^T(z - x) \leq 0\}$;
  x := center($\mathcal{E}$);