6.1. Fractals and self-similarity.
6.1.2 Examples of self-similar fractals

- A self-similar object is an object which contains (rough) copies of itself on arbitrarily small scales.

- The Cantor set: each sub-set is an exact scaled-down copy of the whole set:
• The Cantor set contains structure on arbitrarily small scales:
  – for each point in the Cantor set, there exists another point in the set arbitrarily close to it
  – yet all points in the cantor set are disconnected (for each two point in the set, there exist a finite interval of points between them which are *not* in the set)
• The Koch curve
Each segment contains an exact scaled-down copy of itself, with scale 1/3. (Can be rotated)
• The Mandelbrot set
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• Bifurcation diagrams
• Bifurcation diagrams

Tent Map bifurcation diagram
• Fractals in biology
• Coastlines...
• Random Iterated Systems…
6.2 Fractal dimension
6.2.1 Why is there a problem?

- Mandelbrot: “What is the length of the coastline of Britain?”

  \[
  \begin{align*}
  S=3, & \quad L<2 \\
  S=2, & \quad L=3 \\
  S=1, & \quad L=7 \\
  S=1/2, & \quad L=20
  \end{align*}
  \]

- Does the idea of “total length” make sense? How do we measure it?
Example of the Koch curve

Length = L

⇒

Length = {?}
Example of the Koch curve

Length = L

Length = (4/3) L

Length = 3L

Length = 3 (4/3) L
Example of the Koch curve

\[ 3 \left( \frac{4}{3} \right)^n L \quad \text{for } n = 1, 2, 3, \ldots \]

Total length of Koch curve? INFINITY!
Total area enclosed? \( < \pi L^2 \)
6.2.2 Definition of topological dimension.

- A set S has topological dimension k if each point in the set can be enclosed in an arbitrarily small neighborhood whose boundaries intersect S in a set of dimension k-1, and k is the smallest integer for which this holds.

- If k = 0, then the small neighborhoods must not intersect the set S.
• Examples!
  – A finite set of points: topological dimension 0

Arbitrarily small neighborhood of any point can be chosen such as to never intercept the set.
• Examples!
  – A line/segment: topological dimension 1

Small neighborhood of a point always intersects the set in a finite set of points (here, 1 or 2), which has dimension 0 (see previously)
• Examples!
  – A plane/any other area: topological dimension 2

Small neighborhood of a point always intersects the set in a line (here, a small circle) which has dimension 1.
• The Cantor set: topological dimension?

• The Koch curve: topological dimension?
6.2.3 Definition of fractal dimension.

- Several possible definitions of fractal dimension (not always in agreement).
- For self-similar fractals, we use the similarity dimension.

Definition: Given a self-similar set composed of \( N \) copies of itself scaled down by a factor \( r \), then the similarity dimension is

\[
d = \frac{\ln N}{\ln r}
\]
• Examples!
  – A square

  $r = 2, \ N = 4$
  $r = 3, \ N = 9$
  $r = 4, \ N = 16$

  For the square, $N \propto r^2$

  $d = \frac{\ln N}{\ln r} = 2$

  Fractal dimension = topological dimension: this is NOT a fractal!
• Self similar dimension of a line?

• Self-similar dimension of a cube?
• Self-similar dimension of the Cantor set:

\[ d = \]

• Self-similar dimension of the Koch curve:

\[ d = \]
• Note:
  – Fractal dimensions are not integers!
  – this definition of dimension only works for self-similar fractals. However not all fractals are self-similar
  – Other definitions of dimensions also exist (see Strogatz, Devaney)…
  – Some fractals still have unknown fractal dimension

• Fractals are beautiful objects! For lots of examples, see “The Beauty of Fractals” (Peitgen & Richter).
• More to come next lectures…