Game interactions and dynamics on networked populations

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The replicator equation on graphs

Main idea behind Evolutionary game theory

**Infinite** population of **identical** players (same payoff matrix), but **preprogrammed** with different pure strategies

A couple is randomly drawn to play a game

Fittest players are able to produce more offspring
Evolutionary game theory

- Infinite population of equal individuals (players or replicators)
- Each individual exhibits a certain phenotype (= it is preprogrammed to play a certain strategy).
- An individual asexually reproduces himself by replication. Offspring will inherit the strategy of parent.

Nature favors the fittest

- Two replicators are randomly drawn from the population. Only one of them can reproduce himself.
- They play a game in which strategies are the phenotypes.
- Fitness of a replicator is evaluated like for game theory!
- The fittest is able to reproduce himself. The share of population with that phenotype will increase.

Natural selection!
The replicator equation on graphs

Replicator equation

How to describe the evolution of the pure strategies distribution over the whole population?

\[ \dot{x}_s = x_s(p_s(x) - \phi(x)) \]

- \( x_s \) is the share of population with strategy \( s \)
  - \( 0 \leq x_s \leq 1, \ \sum_s x_s = 1 \ \forall t \geq 0 \)
- \( p_s(x) = e_s^T Bx \) is the payoff obtained by a player with strategy \( s \)
- \( \phi(x) = x^T Bx \) is the average payoff obtained by a player
- \( B \) is the payoff matrix

- All Nash equilibria are rest points.
- A Lyapunov stable rest point is always a Nash equilibrium.


Marginal stability and existence of infinite equilibria are also possible, e.g. with payoff matrices \([a \ b; a \ b]\).
...but in real world situations, things are different!

- Finite population of individuals
- Population has a structure: an individual can meet only his neighbors
- Individuals are described by vertices of a graph, edges are the connections between them. $A = \{a_{v,w}\}$ is the adjacency matrix of the graph
- In general, players aren’t preprogrammed to play a pure strategy; they can be ”mixed” (i.e. they can be partially cooperator and partially defector, selfish and altruist...)

**Efforts to extend the replicator equation to networks**

- The replicator equation on graphs has been obtained for networks with infinite vertices
- Algorithmic approaches have been used to consider a finite population evolving in discrete time
How can we study the behavior of a finite population of different players organized on a network?
Our proposal for a replicator equation on graphs

**Main idea** A mixed player is an infinite population of replicators!

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Step 1. From the RE to the multipopulation RE

a) The population is well mixed and the individuals differ only for the chosen pure strategy

b) The population is splitted in subpopulations with different characteristics (not the strategies)

c) The subpopulations are organized as the vertices of a fully connected graph
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Modeling phase: preliminaries

- $x_{v,s}$ is the share of replicators in subpopulation $v$ with strategy $s$
- $x_v = [x_{v,1} \ldots x_{v,M}]^T$ is the strategy distribution of the subpopulation $v$
- $p_{v,s} = \pi_v(e_s, x_{-v})$ is the average payoff earned by a replicator of subpopulation $v$ with strategy $s$ ($e_s$ is the $s$-th standard versor of $\mathbb{R}^M$)
- $\phi_v = \pi_v(x_v, x_{-v}) = \sum_{s=1}^M x_{v,s} p_{v,s}$ is the average payoff earned by a replicator of subpopulation $v$

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$^1$In general, $\pi_v(x_v, x_{-v})$ is the payoff earned by the mixed player $v$ when he plays strategy $x_v$. 
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The dynamics of interactions (1 / 3)

- $\tau$: interval between interactions
- $n_v(t)$: number of replicators inside subpopulation $v$
- $p_{v,s}(t)$: fitness of a replicator which uses $s$ in subpopulation $v$ at $t$
- $n_v(t)\chi_{v,s}(t)p_{v,s}(t)\tau$: number of offsprings produced by the replicators with strategy $s$ in $v$
- $n_v(t)\chi_{v,s}(t)(1 + p_{v,s}(t)\tau)$: number of replicators (parents and offsprings) with strategy $s$ in $v$ at $t + \tau$
- $n_v(t + \tau) = n_v(t)(1 + \phi_v(t)\tau)$: subpopulation size at $t + \tau$
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The dynamics of interactions (2 / 3)

What happens to $x_{v,s}(t + \tau)$?

$$x_{v,s}(t + \tau) = \frac{n_v(t)x_{v,s}(t)(1 + p_{v,s}(t)\tau)}{n_v(t + \tau)}$$

$$= \frac{x_{v,s}(t)(1 + p_{v,s}(t)\tau)}{1 + \phi_v(t)\tau}$$

Then:

$$\frac{x_{v,s}(t + \tau) - x_{v,s}(t)}{\tau} = \frac{x_{v,s}(t)(p_{v,s}(t) - \phi_v(t))}{1 + \phi_v(t)\tau}$$

When $\tau$ goes to 0...

$$\dot{x}_{v,s}(t) = x_{v,s}(t)(p_{v,s}(t) - \phi_v(t))$$
The replicator equation on graphs

The dynamics of interactions (3 / 3)

\[ \dot{x}_{v,s}(t) = x_{v,s}(t)(p_{v,s}(t) - \phi_v(t)) \]

- When the strategy \( s \) is fitter than the average, then the share of population that uses \( s \) grows; specifically, when \( p_{v,s}(t) > \phi_v(t) \), then the variable \( x_{v,s}(t) \) increases, and decreases otherwise.

Multipopulation replicator equation

Step 2. The multipopulation RE on graphs

- The subpopulations are organized as the vertices of a generic finite graph and each vertex behaves as a population player.
- The replicators of a vertex play 2-players games with the replicators of all the other vertices connected to him.

Calculate the payoffs of the vertex players.
Step 3. Calculate the payoffs (1/3)

- A vertex player sees all his neighbors as an "average player"
- This game is equivalent to a N-players M-strategies game
Step 3. Calculate the payoffs (2/3)

The payoffs depend on the adjacency matrix \( A = \{a_{v,w}\} \) of the graph.

- Payoff obtained by player \( v \) when players use strategies \( s_1, \ldots, s_N \) is a tensor:
  \[
  \bar{\pi}_G^v(s_1, \ldots, s_N) = e_{s_v}^T B_v \left( \frac{1}{d_v} \sum_{v=1}^{N} a_{v,w} e_{s_w} \right)
  \]

- When strategy distributions \( x_v \) are known for each vertex player, then the average payoff can be evaluated as follows:
  \[
  \pi_G^v = \sum_{s_1=1}^{M} \ldots \sum_{s_N=1}^{M} \left( \prod_{w=1}^{N} x_{w,s_w} \right) \bar{\pi}_G^v(s_1, \ldots, s_N).
  \]

Jackson M. O., Zenou Y., 2014. Games on Networks, Handbook of Game Theory, Vol. 4, Peyton Young and Shmuel Zamir, eds., Elsevier Science
Step 3. Calculate the payoffs (3/3)

The extension to the mixed strategy set allows to evaluate the following payoff functions:

- Payoff obtained by player $v$ with strategy $s$:
  \[ p_{v,s}^G = e_s^T B_v k_v \]

- Average payoff obtained by player $v$:
  \[ \phi_v^G = x_v^T B_v k_v \]

where

\[ k_v = \frac{1}{d_v} \sum_{v=1}^{N} a_{v,w} x_w \]
Multipopulation RE + suitable payoffs = RE on graphs

\( x_{v,s} \) is the probability that player \( v \) will play strategy \( s \)

or

the probability that a replicator inside \( v \) is preprogrammed to use strategy \( s \). Then, the replicator equation on graphs (RE-G) is:

\[
\dot{x}_{v,s} = x_{v,s}(p^G_{v,s} - \phi^G_v)
\]

The Cauchy problem:

\[
\begin{cases}
\dot{x}_{v,s} = x_{v,s}(p^G_{v,s} - \phi^G_v) \\
x_{v,s}(t = 0) = c_{v,s}
\end{cases}
\] \quad \forall v \in V, \; \forall s \in S,
Properties of RE-G: invariance and pure stationary points

- **Simplex invariance**
  \[ x_v(0) \in \Delta_M \quad \forall v \Rightarrow x_v(t) \in \Delta_M \quad \forall v, \forall t > 0. \]

- **Pure strategies are stationary points**
  If \( x_v = e_s \), then \( \phi_v^G = p_v^{G,s} \),  
  \[ \dot{x}_v,s = x_v,s(p_v^{G,s} - \phi_v^G) = 0. \]
  Moreover,  
  \[ \dot{x}_v,r = x_v,r(p_v^{G,r} - \phi_v^G) = 0, \quad \text{since } x_v,r = 0, \text{ for all } r \neq s. \]
  Then \( x_v \) is a stationary point of the equation.
Properties of RE-G: Evolutionary stable strategies

- *Evolutionary stable strategy profile* - The idea of evolutionary stable strategy profile in the multipopulation context is slightly different from the standard case. Specifically, \( \{x_1, \ldots, x_N\} \) is an evolutionary stable strategy profile if, for every strategy profile \( \{y_1, \ldots, y_N\} \neq \{x_1, \ldots, x_N\} \) there exists some \( \bar{\epsilon}_y \) such that for all \( \epsilon \in (0, \bar{\epsilon}_y) \), and with the strategy profile \( \{z_1, \ldots, z_N\} \), defined for each \( v \in \{1, \ldots, N\} \) as 
\[
z_v = \epsilon y_v + (1 - \epsilon)x_v,
\]
we have \( \pi_w(x_w, z_{-w}) > \pi_w(y_w, z_{-w}) \) for some \( w \in \{1, \ldots, N\} \).

- An evolutionary stable strategy profile \( \{x_1, \ldots, x_N\} \) is also strict Nash equilibrium of the underlying game, and vice versa.
Properties of RE-G: equivalence to the standard RE

**Theorem**

Let $X(t)$ be the unique solution of the Cauchy problem RE-G, where $x_{v,s}(0) = c_s \ \forall v$ and $B_v = B$. Moreover, let $y(t)$ be the unique solution of the standard Cauchy problem with $y_s(0) = c_s$. Then, $x_v(t) = y(t) \ \forall v, \forall t \geq 0$. 


Properties of RE-G: stationary points, NE and ESS

Let \( \{x_1, \ldots, x_N\} \) be a stationary point of RE-G.

- If \( x_{v,s} > 0 \ \forall v, s \), then \( \{x_1, \ldots, x_N\} \) is a Nash equilibrium of underlying game.
- If \( \{x_1, \ldots, x_N\} \) is Lyapunov stable stationary point of RE-G, then it is a Nash equilibrium.

- Evolutionary stable strategy profiles are asymptotically stable stationary points of RE-G, and viceversa.
The replicator equation on graphs

The RE-G for \((N, 2)\)-games

The payoff matrix for 2-strategies games is:

\[
\mathbf{B}_v = \begin{bmatrix}
    b_{v,1,1} & b_{v,1,2} \\
    b_{v,2,1} & b_{v,2,2}
\end{bmatrix}.
\]

Let \(\sigma_{v,1} = b_{v,1,1} - b_{v,2,1}\), \(\sigma_{v,2} = b_{v,2,2} - b_{v,1,2}\) and \(k_v(y) = [k_{v,1}(y) \ k_{v,2}(y)]^T\), then the RE-G becomes:

\[
\dot{y}_v = y_v(1 - y_v) f_v(y),
\]

where

\[
f_v(y) = \sigma_{v,1} k_{v,1}(y) - \sigma_{v,2} k_{v,2}(y).
\]
The replicator equation on graphs

Stationary points of RE-G for \((N, 2)\)-games

The set of stationary points is:

\[ \Theta^* = \left\{ y^* \in [0, 1]^N : \forall v \ y_v^* = 0 \lor y_v^* = 1 \lor f_v(y^*) = 0 \right\} . \]

1. **Pure stationary points** The set \(\Theta^p = \{0, 1\}^N\) is a subset of \(\Theta^*\). Pure stationary points always exist, for every adjacency matrix \(A\) and payoff matrices \(B_v\).

2. **Mixed stationary points** The set \(\Theta^m = (0, 1)^N \cap \Theta^*\) is a subset of \(\Theta^*\). Equivalently,

\[ \Theta^m = \left\{ y^* \in (0, 1)^N : \sigma_{v,1} k_{v,1}(y^*) = \sigma_{v,2} k_{v,2}(y^*) \ \forall v \right\} . \]

3. **Pure/Mixed stationary points** The set \(\Theta^{mp} = \Theta^* \setminus (\Theta^p \cup \Theta^m)\) is a subset of \(\Theta^*\).
**Theorem**

Let $\mathbf{y}^* \in \Theta^p$. Then, the following statements are equivalent:

(a) $((\sigma_{v,1} k_{v,1}(\mathbf{y}^*) \leq \sigma_{v,2} k_{v,2}(\mathbf{y}^*) \land y_v^* = 0) \lor (\sigma_{v,1} k_{v,1}(\mathbf{y}^*) \geq \sigma_{v,2} k_{v,2}(\mathbf{y}^*) \land y_v^* = 1)) \ \forall v$

(b) $\lambda_v(J(\mathbf{y}^*)) \leq 0 \ \forall v$

(c) $\mathbf{y}^* \in \Theta^{NE}$

**Corollary**

Let $\mathbf{y}^* \in \Theta^p$. Then, the following statements are equivalent:

(a') $((\sigma_{v,1} k_{v,1}(\mathbf{y}^*) < \sigma_{v,2} k_{v,2}(\mathbf{y}^*) \land y_v^* = 0) \lor (\sigma_{v,1} k_{v,1}(\mathbf{y}^*) > \sigma_{v,2} k_{v,2}(\mathbf{y}^*) \land y_v^* = 1)) \ \forall v$

(b') $\lambda_v(J(\mathbf{y}^*)) < 0 \ \forall v$

(c') $\mathbf{y}^* \in \Theta^{NES}$
Previous theorems provide an interpretation of the stability of stationary points of RE-G. Indeed,

- $\sigma_{v,s}$, $s = 1, 2$, are the gains obtained when choosing pure strategy 1 or 2.
- $k_{v,s}(y^*)$, $s = 1, 2$, are related to the number of players connected to vertex $v$.

When for all vertices $v$s, the gain of neighbor players which choose the same pure strategy as player $v$ is bigger, then the stationary state is stable.
The replicator equation on graphs

$(N, 2)$-games: stability of mixed stationary points

**Theorem**

Suppose that $\sigma_{v,1} = \sigma_1$ and $\sigma_{v,2} = \sigma_2 \ \forall v$ with $\text{sign}(\sigma_1) = \text{sign}(\sigma_2) \neq 0$. Then there always exists a stationary point $y^* \in \Theta^m$, such that $y^*_v = \frac{\sigma_2}{\sigma_1 + \sigma_2} \ \forall v$. Moreover, $\lambda(J(y^*)) = \frac{\sigma_1 \sigma_2}{\sigma_1 + \sigma_2} \lambda(A)$.  

- If $A$ is invertible, then the mixed stationary state $y^*$ is unique and does not depend on $A$ itself.
- If $A$ is not invertible, then there exist other stationary points which depend on $A$.
- The linearized stability of $y^*$ depends on the eigenvalues of the adjacency matrix $A$. 


Replicator equation on graphs: simulation results

**Bistable payoff matrix:** \( B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \)

Since all vertices are the same at initial time, then evolution is equivalent to classical replicator equation.
Replicator equation on graphs: simulation results

Bistable payoff matrix: \( B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \)

Network is resistant to the mutator.
The replicator equation on graphs: simulation results

Bistable payoff matrix: \( B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \)

The mutator is stronger thanks to his connections. We have a diffusive process and coexistence of strategies.
The replicator equation on graphs

Bistable payoff matrix: \( B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \)

The central mutator is helped by another mutator. They impose their strategy to all the population!
Replicator equation on graphs: simulation results

**Prisoners’ dilemma game:**

\[
B = \begin{bmatrix}
1 & 0 \\
1.5 & 0 \\
\end{bmatrix}
\]

**Resilience of cooperation in non strict Prisoners’ dilemma**
The replicator equation on graphs

Replicator equation on graphs: simulation results

Attractive mixed Nash equilibrium: \( B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \)

The attractive mixed Nash equilibrium of RE is not homogeneously stable in the RE-G
Conclusions and Future Work

The theoretical formulation of the RE-G provides a new way for
- describing the dynamical interactions among individuals in a society
- obtaining a general framework to derive suitable equations of such interactions
- understanding the properties of unexplained real world phenomena

Future work will investigate
- complex graph topologies and their influence on the system dynamics: properties of the adjacency matrix $A$
- equivalence of the model with infinite dimensional systems when the number of vertices becomes high
- asymptotic stability of equilibria of the ordinary differential equations by means of nonlinear techniques
- Network reconstruction by solving suitable inverse problems