CORRIGENDUM
NON-TRIVIAL NON-NEGATIVE PERIODIC SOLUTIONS OF A SYSTEM OF DOUBLY DEGENERATE PARABOLIC EQUATIONS WITH NONLOCAL TERMS

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Abstract. We correct a flaw in the proof of [1, Lemma 2.3].

1. Corrigendum. This Corrigendum concerns the proof of [1, Lemma 2.3]. In that proof there is a flaw in the estimate of \( \log x_k \) due to an incorrect inequality. We provide here a correct estimate of \( \log x_k \) in 1.5 which preserves the validity of Lemma 2.3. For the reader convenience we recall the statement of Lemma 2.3 and we give its complete proof. Here \( m > 1 \) and \( p > 2 \).

Lemma 2.3 Let \( K > 0 \) and assume that \( u \) is a non-negative periodic function such that \( u \in C(\mathcal{Q}_T), u^m \in L^p(0,T;W^{1,p}_0(\Omega)) \) and satisfying

\[
u_t - \text{div}\{[|\nabla (u^m + \epsilon u)|^2 + \eta]^\frac{p-2}{2} \nabla (u^m + \epsilon u)} \leq Ku, \quad \text{in } \mathcal{Q}_T
\]

and \( u(\cdot, t)|_{\partial \Omega} = 0, \) for \( t \in [0,T] \). Then there exists \( R > 0 \) and independent of \( \epsilon \) and \( \eta \) such that

\[\|u\|_{L^\infty} \leq R.\]

Proof. We follow Moser’s technique to show the stated a priori bounds. Multiplying

\[
u_t - \text{div}\{[|\nabla (u^m + \epsilon u)|^2 + \eta]^\frac{p-2}{2} \nabla (u^m + \epsilon u)} \leq Ku
\]

by \( u^{s+1} \), with \( s \geq 0 \), integrating over \( \Omega \) and passing to the limit as \( h \to 0 \) in the Steklov averages \( u_h \) we have

\[
K\|u(t)\|_{L^{\frac{s+2}{s+1}}(\Omega)} \geq \frac{1}{s + 2} \frac{d}{dt}\|u(t)\|_{L^{\frac{s+2}{s+1}}(\Omega)}^s + (s + 1) \int_\Omega \|\nabla (u^m + \epsilon u)|^2 + \eta\|\frac{p-2}{2} (mu^{m-1} + \epsilon)u^s |\nabla u|^2.
\]

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Since $p > 2$, $m > 1$ and
\[ u^{(m-1)(p-2)} |\nabla u|^{p-2} \leq (mu^{m-1} + \epsilon)^{p-2} |\nabla u|^{p-2} \leq |\nabla (um + \epsilon u)|^2 + \eta \frac{s + 2}{s + 1}, \]
we have
\[ \frac{1}{s + 2} \frac{d}{dt} \|u(t)\|^{s+2}_{L^{s+2}(\Omega)} + \int_{\Omega} u^{(p-1)(m-1)+s} |\nabla u|^p \leq K \|u(t)\|^{s+2}_{L^{s+2}(\Omega)}. \]
This implies
\[ K(s + 2) \|u(t)\|^{s+2}_{L^{s+2}(\Omega)} \geq \frac{d}{dt} \|u(t)\|^{s+2}_{L^{s+2}(\Omega)} + \frac{s + 2}{[m(p-1) + s + 1]p} \int_{\Omega} \|u\|^{m(p-1)+s+1}_{L^{p}(\Omega)} \].

For $\epsilon$ and $\eta$ fixed and $k = 1, 2, \ldots$, setting
\[ s_k := 2p^k + \frac{p^k - p}{p - 1} + m - 1, \quad \alpha_k := \frac{p(s_k + 2)}{m(p - 1) + s_k + 1}, \quad w_k := u \frac{m(p-1)+s_k+1}{p}, \]
we obtain by 1.1
\[ \frac{d}{dt} \|w_k(t)\|^{s_k}_{L^{s_k}(\Omega)} + \frac{s_k + 2}{[m(p-1) + s + 1]p} \|\nabla w_k(t)\|^{p}_{L^{p}(\Omega)} \leq K(s_k + 2) \|w_k(t)\|^{s_k}_{L^{s_k}(\Omega)}. \]

Observe that since $s_k \to +\infty$, as $k \to +\infty$, there exists $k_0$ such that $\alpha_k \in (1, p)$ for all $k \geq k_0$. By the interpolation and the Sobolev inequalities, it results
\[ \|w_k(t)\|_{L^{\alpha_k}(\Omega)} \leq \|w_k(t)\|^{\theta_k}_{L^1(\Omega)} \|w_k(t)\|^{1-\theta_k}_{L^\alpha(\Omega)} \leq C \|w_k(t)\|^{\theta_k}_{L^1(\Omega)} \|\nabla w_k(t)\|^1_{L^\alpha(\Omega)} \]
for all $k \geq k_0$. Here $\theta_k = (s - \alpha_k)/(\alpha_k(s - 1))$, $s > p$ is fixed (say $s = p^* + \eta$ if $p < n$, where $p^* := np/(n-p) + C$ is a positive constant. Using the fact that
\[ \|w_k(t)\|_{L^1(\Omega)} = \|w_{k-1}(t)\|^{\alpha_k-1}_{L^{\alpha_k}(\Omega)} \]
and defining $x_k := \sup_{t \in \mathbb{R}} \|w_k(t)\|_{L^{\alpha_k}(\Omega)}$, one has
\[ \|w_k(t)\|^{\alpha_k}_{L^{\alpha_k}(\Omega)} \leq C \|w_{k-1}(t)\|^{\alpha_k-1}_{L^{\alpha_k}(\Omega)} \|\nabla w_k(t)\|^{1}_{L^\alpha(\Omega)} \]
\[ \leq C x_{k-1}^{\alpha_k-1} \|\nabla w_k(t)\|^{1}_{L^\alpha(\Omega)}, \]
for all $k \geq k_0$. Thus, by 1.2,
\[ \frac{d}{dt} \|w_k(t)\|^{\alpha_k}_{L^{\alpha_k}(\Omega)} \leq K(s_k + 2) \|w_k(t)\|^{s_k}_{L^{s_k}(\Omega)} - C \|w_k(t)\|^{s_k}_{L^{s_k}(\Omega)} x_{k-1}^{\alpha_k-1} \|\nabla w_k(t)\|^{s_k}_{L^{s_k}(\Omega)} \]
\[ = \left( k - \frac{C}{[m(p-1) + s_k + 1]p} \|w_k(t)\|^{s_k}_{L^{s_k}(\Omega)} x_{k-1}^{\alpha_k-1} \right) \cdot (s_k + 2) \|w_k(t)\|^{s_k}_{L^{s_k}(\Omega)}, \]
for all $k \geq k_0$. By Lemma 1.1 below, the differential inequality 1.3 implies
\[ \|w_k(t)\|_{L^{\alpha_k}(\Omega)} \leq \left( \frac{K}{M_k} x_{k-1}^{\alpha_k-1} \right)^{\eta_k}, \]
for all $k \geq k_0$, where $\eta_k := (1 - \theta_k)/[p - \alpha_k(1 - \theta_k)]$ and $M_k := C/[m(p-1) + s_k + 1]p$. By definition of $x_k$ and 1.4 we get
\[ x_k \leq \left( \frac{K}{M_k} \right)^{\eta_k} x_{k-1}^{\alpha_k-1}. \]
for all $k \geq k_0$, with $\nu_k := p\alpha_{k-1}\theta_k/[p - \alpha_k(1 - \theta_k)]$.

If $x_{k-1} \leq 1$, using the fact that $x_{k-1} = \sup_{t \in \mathbb{R}} \|u(t)\|_{s_{k-1}+2}^{m(p-1)+s_{k-1}+1}$, one has $\|u\|_{L^\infty} \leq 1$. Now, assume $x_{k-1} > 1$ and observe that there exists $\bar{k}_0$ such that, for all $k \geq \bar{k}_0$, $\eta_k \leq 1/(p\theta)$ and $\nu_k \leq p$. Here $\theta := (s - p)/(p(s - 1))$. Without loss of generality, assume $k_0 = \max\{\bar{k}_0, k_0\}$. Then, there exists a positive constant $A$ such that

$$x_k \leq \left(\frac{K}{C}\right)_{\eta_k} [m(p-1) + s_k + 1]_{p\nu_k} x_k^{\nu_k}$$

$$\leq \left(\frac{K}{C}\right)_{\eta_k} \left(\mp + \frac{2p^{k+1} + s_k}{p - 1}\right)_{p\nu_k} x_k^{\nu_k}$$

$$\leq A p^{k+1} x_k^{\nu_k}$$

for all $k \geq k_0$. Thus

$$\log x_k \leq \log A + \frac{k + 1}{\theta} \log p + p \log x_{k-1}$$

$$\leq \log A \sum_{i=0}^{k-k_0-1} p^i + \frac{\log p}{\theta} \sum_{i=k_0+1}^{k+1} i p^{k+1-i} + p^{k-k_0} \log x_{k_0}$$

$$\leq \log A \left(\frac{p}{(p-1)^2}\right)_{k_0(p-1) + 2p - 1}$$

$$+ \log A \frac{1 - p^{k-k_0}}{1 - p} + p^{k-k_0} \log x_{k_0}.$$  

Indeed, taking $x = \frac{1}{p}$ in $x \frac{d}{dx} \sum_{i=0}^{k+1} x^i = x \frac{d}{dx} \left(\frac{1 - x^{k+2}}{1 - x}\right)$, it results

$$\sum_{i=k_0+1}^{k+1} i p^{k+1-i} = \frac{p^{k+3}}{(p - 1)^2} \left[\frac{1}{p} (k + 1) - k - 2\right] - \frac{1}{p^{k_0+2}} \left(k_0 + 1\right) \frac{1}{p} - k_0 - 2\right]$$

$$\leq \frac{p^{k+3}}{(p - 1)^2} \frac{1}{p} \left(k_0 + 2 - k_0 + 1\right) = \frac{p^{k-k_0}}{(p - 1)^2} \left(k_0(p-1) + 2p - 1\right).$$

Then, by 1.5, it follows

$$x_k \leq A \frac{p^{k-k_0}}{(p - 1)^2} p^{k-k_0} x_{k_0}^{\nu_k}.$$ 

Since $x_k = \sup_{t \in \mathbb{R}} \|u(t)\|_{s_{k}+2}$, we obtain

$$\|u(t)\|_{L^\infty} \leq \lim_{k \to \infty} \|u(t)\|_{s_{k}+2}$$

$$\leq \lim_{k \to \infty} \left\{A \frac{p^{k-k_0}}{(p - 1)^2} p^{k-k_0} x_{k_0}^{\nu_k} \frac{p^{k-k_0}}{(p - 1)^2} p^{k-k_0} x_{k_0}^{\nu_k} \frac{p^{k-k_0}}{(p - 1)^2} p^{k-k_0} x_{k_0}^{\nu_k} \frac{p^{k-k_0}}{(p - 1)^2} p^{k-k_0} x_{k_0}^{\nu_k} \right\}$$

$$=: R, \quad \forall t \in \mathbb{R},$$

where $R$ is a positive constant. Hence $\sup_{t \in \mathbb{R}} \|u(t)\|_{L^\infty} \leq R$. It remains to prove that $R$ is independent of $\epsilon$ and $\eta$ as claimed. To this aim it is sufficient to prove that there exists $C > 0$ such that $x_{k_0} \leq C$. Indeed, by the inequality 1.1 with $s_0 := s_{k_0},$
it follows
\[
\frac{d}{dt} \|u(t)\|_{L^{p+2}(\Omega)}^{p+2} + \frac{s_0 + 2}{m(p - 1) + s_0 + 1} \int_{\Omega} \left| \nabla u \right|^{m(p-1)+s_0+1} \leq K(s_0+2) \|u(t)\|^{s_0+2}_{L^{p+2}(\Omega)}.
\]

Moreover, by the Hölder inequality with \( r := \frac{m(p-1)+s_0+1}{s_0+2} \) and the Poincaré inequality, we have
\[
\|u(t)\|_{L^{p+2}(\Omega)}^{p+2} \leq C \|\nabla u\|_{L^{p}(\Omega)}^{m(p-1)+s_0+1}
\]
for a positive constant \( C \). Thus, using 1.6, one has
\[
\frac{d}{dt} \|u(t)\|_{L^{p+2}(\Omega)}^{p+2} + \frac{s_0 + 2}{m(p - 1) + s_0 + 1} \|u(t)\|_{L^{p+2}(\Omega)}^{m(p-1)+s_0+1} \leq K(s_0+2) \|u(t)\|_{L^{p+2}(\Omega)}^{s_0+2}.
\]
Hence
\[
\frac{d}{dt} \|u(t)\|_{L^{p+2}(\Omega)}^{s_0+2} \leq \|u(t)\|_{L^{p+2}(\Omega)}^{s_0+2} \left(K(s_0+2) - M \|u(t)\|_{L^{p+2}(\Omega)}^{m(p-1)+s_0+1}\right),
\]
where \( M := \frac{s_0 + 2}{C(m(p - 1) + s_0 + 1)^{m(p-1)+s_0+1}} \). Lemma 1.1 implies
\[
\|u(t)\|_{L^{p+2}(\Omega)} \leq \left\{ CK(m(p - 1) + s_0 + 1)^{m(p-1)+s_0+1}\right\} \frac{1}{m(p-1)+s_0+1}, \quad \forall t \in \mathbb{R}.
\]
Thus there exists \( C > 0 \) such that \( x_{\epsilon, \eta} = \sup_{t \in \mathbb{R}} \|u(t)\|_{s_0+2} \leq C \), as claimed.

\[\square\]

**Lemma 1.1.** Let \( f : \mathbb{R} \to (0, +\infty) \) be a differentiable and \( T \)-periodic function; suppose that there exist positive constants \( s, \alpha, \beta, \gamma \) such that
\[
f'(t) \leq f'(t)(\beta - f^\alpha(t)),
\]
for all \( t \in \mathbb{R} \). Then \( \beta - \gamma f^\alpha(t) \geq 0 \) for all \( t \in \mathbb{R} \).

We took advantage of this occasion to provide also an explicit estimate of \( x_{\epsilon, \eta} \) independent of \( \epsilon \) and \( \eta \), which shows that \( R \) is independent of these parameters.

We finally point out some misprints and imprecisions that could mislead the reader: at page 39, the uniqueness of the solution of (3) follows from [2, Theorem 32D], and Lemma 2.2 is proved by using [19, Theorem 1.2]; at the end of p. 40 the right equation for \( z \) is
\[
L_{\epsilon, \eta, p}^m[z] + Mz = 0
\]

**REFERENCES**


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